

Last time: spectral seq. of a double complex

$$E_2^{p,q} = H_{\text{horz}}^p (H_{\text{vert}}^q) \Rightarrow H^{p+q} (\text{total complex})$$

Worksheet: Tor is balanced.

Now: Ext is balanced.

that is: $\text{Ext}_R^i(M, N)$ can be computed either with a proj. res of M or an inj. res of N

take a proj. res $\dots \rightarrow P^{-2} \rightarrow P^{-1} \rightarrow P^0 \rightarrow M \rightarrow 0$
and an inj. res $0 \rightarrow N \rightarrow I^0 \rightarrow I^1 \rightarrow I^2 \rightarrow \dots$

form the double complex $C^{p,q} = \text{Hom}(P^{-p}, I^q)$

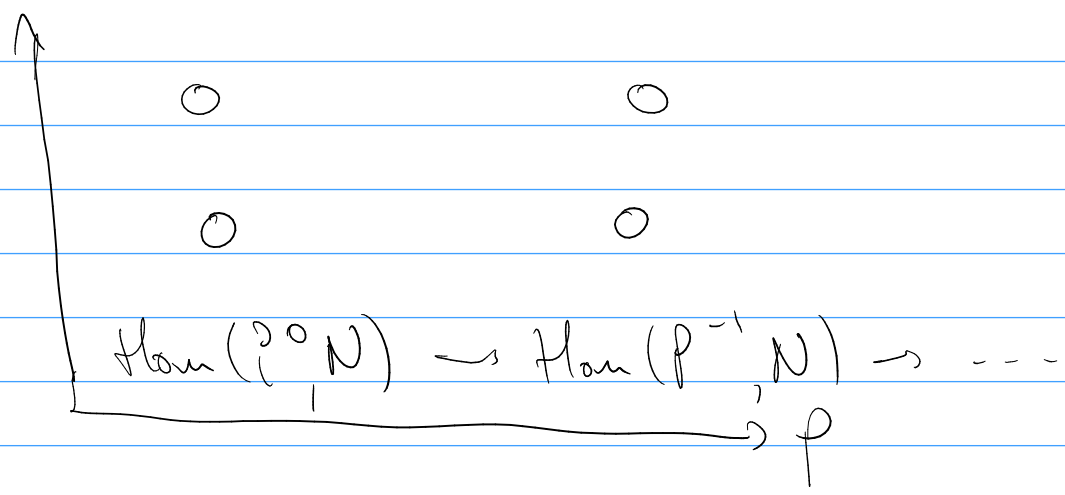
$$\begin{array}{ccc} \uparrow & & \\ \text{Hom}(P^0, I^2) & \text{Hom}(P^{-1}, I^2) & \\ \uparrow & \uparrow & \\ \text{Hom}(P^0, I^1) & \text{Hom}(P^{-1}, I^1) & \\ \uparrow & \uparrow & \\ \text{Hom}(P^0, I^0) & \text{Hom}(P^{-1}, I^0) & \text{etc} \\ \leftarrow & \rightarrow & P \end{array}$$

columns are $\text{Hom}(\text{fixed } P, \text{inj res of } N)$

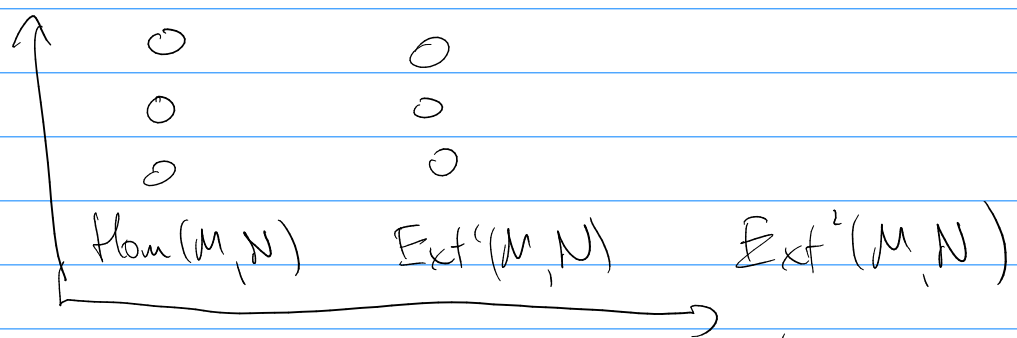
take vertical coho.

because P is projective, $\text{Hom}(P, -)$ commutes with coho.

so E^1 page looks like



and $E_{p,q}^2$ looks like



where Exts are computed w/ proj res of M .

so $H^i(\text{total complex}) = \text{Ext}^i(M, N)$ computed w/ proj res.

transport every thing.

now $C^{p,q} = \text{Hom}(P^{-q}, I^p)$

spectral seq will show that

$H^i(\text{total complex}) = \text{Ext}^i(M, N)$ computed w/ inj. res of N .

but two total complexes are isomorphic. \square

Computing Ext and Tor with
any old res, not proj. or inj.

$$\text{let } \cdots \rightarrow A^{-2} \rightarrow A^{-1} \rightarrow A^0 \rightarrow N \rightarrow 0 \quad (*)$$

be a resolution by any kind of modules.

if we know more about
 $\text{Tor}(M, A^i)$ or $\text{Ext}(M, A^i)$

than about $\text{Tor}(M, N)$ or $\text{Ext}(M, N)$,

could break (*) up into short exact seqs,
do long exact seq. of each,
and learn about $\text{Tor}(M, N)$ or $\text{Ext}(M, N)$

better:

$$\left[\begin{array}{l} E_1^{p,q} = \text{Tor}_{-q}^p(M, A^i) \Rightarrow \text{Tor}_{p-q}^2(M, N) \\ \text{(third quadrant)} \end{array} \right.$$

$$E_1^{p,q} = \text{Ext}_2^q(M, A^p) \Rightarrow \text{Ext}_2^{p+q}(M, N)$$

(second quadrant — need fin. many A s only)

$$E_1^{p,q} = \text{Ext}_2^q(A^{-p}, M) \Rightarrow \text{Ext}_2^{-p+q}(N, M)$$

(first quadrant)

E_1 page is

$$\begin{array}{c} \begin{array}{c} \dots \rightarrow M \otimes A^1 \rightarrow M \otimes A^0 \\ \rightarrow \text{Tor}_1(M, A^1) \rightarrow \text{Tor}_1(M, A^0) \\ \text{etc} \rightarrow \text{Tor}_2(M, A^0) \end{array} \end{array} \begin{array}{c} \uparrow \\ \uparrow \\ \uparrow \end{array} \begin{array}{c} \rho \\ \rho \\ \rho \end{array}$$

as claimed.