Last time: spectral seq. of a double complex

$$E_2^{p,q} = H^p_{horz}(H^q_{vert}) \Rightarrow H^{p+q}(\text{total complex})$$

Worksheet: Tor is balanced.

Now: Ext is balanced.

that is: $\mathrm{Ext}^i_R(M,N)$ can be computed either with a proj. res of M or an inj. res of N

take a proj res $\cdots \to P^{-2} \to P^{-1} \to P^0 \to M \to 0$

and an inj. res $0 \to N \to I^0 \to I^1 \to I^2 \to \cdots$

form the double complex $C^{p,q} = \mathrm{Hom}(P^{-p}, I^q)$

$$E_0^{p,q} = \begin{array}{ccc} \uparrow q & & \\ & \mathrm{Hom}(P^0, I^2) & \mathrm{Hom}(P^{-1}, I^2) \\ & \uparrow & \uparrow \\ & \mathrm{Hom}(P^0, I^1) & \mathrm{Hom}(P^{-1}, I^1) \\ & \uparrow & \uparrow \\ & \mathrm{Hom}(P^0, I^0) & \mathrm{Hom}(P^{-1}, I^0) \quad \text{etc} \\ & & \to p \end{array}$$

columns are Hom(fixed P, inj res of N)

take vertical coho.

Because P is projective, $\mathrm{Hom}(P, -)$ commutes with coho.

so E^1 page looks like

$$\begin{array}{ccccc}
0 & & 0 & & \\
0 & & 0 & & \\
\mathrm{Hom}(P^0, N) & \longrightarrow & \mathrm{Hom}(P^{-1}, N) & \longrightarrow & \cdots
\end{array} \quad p$$

and $E^2_{p,q}$ looks like

$$\begin{array}{ccc}
0 & 0 & \\
0 & 0 & \\
0 & 0 & \\
\mathrm{Hom}(M,N) & \mathrm{Ext}^1(M,N) & \mathrm{Ext}^2(M,N)
\end{array}$$

where Exts are computed w/ proj res of M.

so $H^i(\text{total complex}) = \mathrm{Ext}^i(M,N)$ computed w/ proj res.

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transpose everything.

now $C^{p,q} = \mathrm{Hom}(P^{-q}, I^p)$

spectral seq will show that

$H^i(\text{total complex}) = \mathrm{Ext}^i(M,N)$ computed w/ inj. res of N.

but two total complexes are isomorphic. $\blacksquare$

Computing Ext and Tor with any old res, not proj. or inj.

let $\dots \to A^{-2} \to A^{-1} \to A^{0} \to N \to 0 \qquad (*)$

be a resolution by <u>any</u> kind of modules.

if we know more about $\mathrm{Tor}(M, A^{\cdot})$ or $\mathrm{Ext}(M, A^{\cdot})$

than about $\mathrm{Tor}(M,N)$ or $\mathrm{Ext}(M,N)$,

could break $(*)$ up into short exact seqs, do long exact seq. of each, and learn about $\mathrm{Tor}(M,N)$ or $\mathrm{Ext}(M,N)$

better:

$$E_1^{p,q} = \mathrm{Tor}^R_{-q}(M, A^p) \Longrightarrow \mathrm{Tor}^R_{p-q}(M,N)$$
(third quadrant)

$$E_1^{p,q} = \mathrm{Ext}_R^q(M, A^p) \Longrightarrow \mathrm{Ext}_R^{p+q}(M,N)$$
(second quadrant — need fin. many As only)

$$E_1^{p,q} = \mathrm{Ext}_R^q(A^{-p}, M) \Longrightarrow \mathrm{Ext}_R^{-p+q}(N,M)$$
(first quadrant)

similarly if $0 \to M \to B^0 \to B^1 \to B^2 \to \cdots$
you get some nice spectral seqs
starting from E_1.

pf of the first one

$\longrightarrow p \qquad \uparrow q$

$$
\begin{tikzcd}
\cdots \arrow[r] & A^{-2} \arrow[r] & A^{-1} \arrow[r] & A^{0} \arrow[r] & N \arrow[r] & 0 \\
& P^{-2,0} \arrow[r] \arrow[u] & P^{-1,0} \arrow[r] \arrow[u] & P^{0,0} \arrow[u] & & \\
& P^{-2,-1} \arrow[r] \arrow[u] & P^{-1,-1} \arrow[r] \arrow[u] & P^{0,-1} \arrow[u] & & \\
& P^{-2,-2} \arrow[r] \arrow[u] & P^{-1,-2} \arrow[r] \arrow[u] & P^{0,-2} \arrow[u] & & \\
& \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] & &
\end{tikzcd}
$$

here is my double complex.
total complex is a proj res of N
as on Friday.

apply $M \otimes_R -$ to get a new double complex.

$$H^i(\text{new total complex}) = \mathrm{Tor}^R_{-i}(M,N)$$

columns of E_0 page compute $\mathrm{Tor}(M,A)$

take vertical homology..

E_1 page is

$$\begin{array}{ll} \cdots \longrightarrow M \otimes A^1 \longrightarrow M \otimes A^0 \\ \cdots \longrightarrow \mathrm{Tor}_1(M, A^1) \longrightarrow \mathrm{Tor}_1(M, A^0) \\ \text{etc} \longrightarrow \mathrm{Tor}_2(M, A^0) \end{array}$$

$q \uparrow \quad \rightarrow p$

as claimed.