At the beginning, I promised:

if $I, J \subset R = k[x_1, \dots, x_n]$ radical ideals

cut out smooth $X, Y \subset \mathbb{A}^n$

let's write $\dim(X \cap Y) = \dim Y - \operatorname{codim} X + k$

Thm: $\operatorname{Tor}^R_{\leq k}(R/I, R/J) \neq 0$

and $\operatorname{Tor}_{>k} = 0$

in particular, $\dim(X \cap Y) =$ expected dim iff $\operatorname{Tor}_{>0}(R/I, R/J) = 0$.

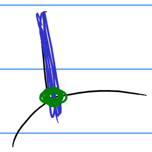
On Wednesday's worksheet, had

$I = (x, y)$ $\quad J = (x, y, z)$ $\quad R = k[x, y, z]$

line $\quad$ a point in $\mathbb{A}^3$

expected dim $= 0 - 2 = -2$

actual dim $= 0$



$\operatorname{Tor}_*(R/I, R/J) = k, k^2, k, 0, \dots$

Outline of end of the course:

know that Tor commutes w/ localization
so we'll prove the thm locally at
each max'l ideal $\underline{m} \subset R$ corresp. to a
point of $X \cap Y$. $(I + J \subset \underline{m})$

① X is smooth iff R/I is a regular ring

② if R is a regular local ring
and R/I is regular then
I is gen'd by a regular seq
whose length $= \dim(R) - \dim(R/I)$
thus R/I is resolved by a koszul cx.

③ let $S = R/J$.

so $\mathrm{Tor}^R_t(R/I, S)$ is computing
the homology of that Kosz. cx. $\otimes S$
$=$ depth of $IS \subset S$

[or call it $I+J/J \subset R/J$ if that's clearer.]

④ S is regular so it's Cohen-Macaulay,
meaning depth of $\underline{m} \subset S = \dim S$
this implies depth $I = \dim S - \dim S/I$
$\forall\ I \subset \underline{m} \subset S$.

So our big thm, translated to alg. language:

if R is a Noeth. local ring and $I \subset \underline{m}$ is gen'd by a regular sequence,

and if $R \to S$ where S is CM,

write $\operatorname{codim} IS = \operatorname{codim} I + k$

then $\operatorname{Tor}^R_{\leq k}(R/I, S) \neq 0$

$\operatorname{Tor}_{>k} = 0$.

R a Noeth. local ring.

<u>depth</u> of $I \subset R$
= length of any maximal reg. seq. in I.
it's measured by homology of a Koszul cx.

(techn: depth of I on the module R.)

the <u>Krull dimension</u> of R is n
if the longest chain of prime ideals is
$p_0 \subset p_1 \subset \cdots \subset p_n \subset R$

the <u>height</u> or <u>codim</u> of $I \subset R$ is n
if the longest chain of primes in I is
$p_0 \subsetneq p_1 \subsetneq \cdots \subsetneq p_n \subseteq I$

prop: $\dim R/I + \operatorname{codim} I \leq \dim R$

pf: chains of primes in R/I
$\longleftrightarrow$ chains of primes $\supseteq I$ in R.

rmk if R is a "catenary" ring,
e.g. a quot. of a regular ring,
then it's $=$ not $<$.

prop let $r \in \mathfrak{m} \subset R$. then

$$\dim R - 1 \leq \dim R/r \leq \dim R$$

and if r is not a zero-div then
$\dim R/r = \dim R - 1$

part of the proof:
take a maximal chain of primes in R/r

this corresponds to

$$r \in \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n \subset R \qquad \text{where } n = \dim R/r.$$

if $\mathfrak{p}_0$ were minimal then r would be a zero-divisor

so if r is not a zero-div
then we can fit one more prime
below $\mathfrak{p}_0$. $\square$

Cor. if $r_1, \dots, r_n \in \mathfrak{m} \subset R$ is a reg. seq.

then $\dim R/(r_1, \dots, r_n) = \dim R - n$.

pf: induction.

main result today: $\forall\ I \subset \mathfrak{m} \subset R$,

$$\operatorname{depth}(I) \le \operatorname{codim} I \le \dim R - \dim R/I$$

prove this $\le$

pf: let $d = \operatorname{depth}(I)$

choose gens for I starting with a reg. seq.

$$\underbrace{r_1, \dots, r_d}_{\text{reg}}, \dots, r_n \in I$$

Consider $S = R/(r_1, \dots, r_d)$ $\quad J = (\bar{r}_{d+1}, \dots, \bar{r}_n) \subset S$

so $S/J \cong R/I$

and $\dim R - \dim R/I = (\dim S + d) - \dim S/J$

$$= d + (\dim S - \dim S/J)$$
$$\ge d + 0.$$