R still a Noeth. local ring.

last time: $\forall\ I \subset R$

$$\operatorname{depth} I \leq \operatorname{codim} \text{ or height } I \leq \dim R - \dim R/I$$

in particular, $\operatorname{depth} \underline{m} \leq \dim R$

__Def__: R is Cohen-Macaulay if $\operatorname{depth} \underline{m} = \dim R$

__Thm.__ if R is CM then every $I \subset R$ has depth $= \dim R - \dim R/I$.

(Pf postponed to Wed)

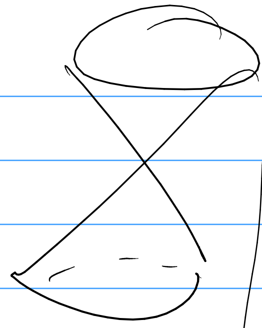
__Examples__

1. if R is regular then it's CM

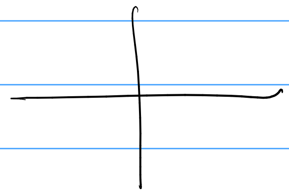
   pf: let $r_1, \dots, r_n$ be a reg. seq that generates $\underline{m}$.
   then $\operatorname{depth}(\underline{m}) = n$
   and $\underbrace{\dim(R/(r_1, \dots, r_n))}_{= 0} = \dim R - n$

2. also R/I when R is regular and $I \subset \underline{m}$ is generated by a reg. seq.
   (local complete intersection, l.c.i)

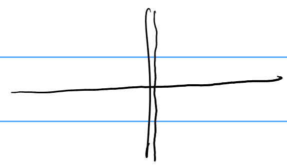
$k[x,y,z]/x^2+y^2-z^2$



$k[x,y]/xy$



$k[x,y]/x^2y$



are CM when localized at any maximal ideal.

$\mathbb{Z}[\sqrt{-3}] = \mathbb{Z}[x]/x^2+3$

loc. at $\underline{m} = (2, 1+\sqrt{-3})$ then it's not reg but it's still CM.

Pf: let $r_1, \ldots, r_k$ be a regular seq. that generates I

extend to a maximal reg seq $r_1 \cdots r_k \cdots r_n \in \underline{m}$
$\hookrightarrow$ maybe these don't generate $\underline{m}$!

by hypothesis, $n = \dim R$

and $\bar{r}_{k+1}, \ldots, \bar{r}_n \in R/I$ are a reg seq. in the maximal ideal $\underline{m}' \subset R/I$

So depth $(\underline{m}') \geq n-k = \dim R/_{I}$.

(in fact we've proved:
$$ {CM}/_{\text{reg seq}} = CM \text{ again} \Bigg) \quad \boxtimes $$

(3) also "determinantal varieties"

= space of $m \times n$ matrices of rank $\leq k$

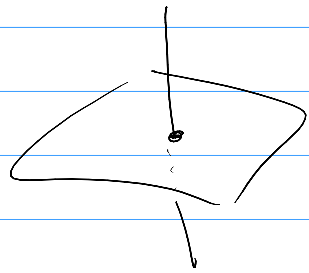
e.g. from WS 16, studied $\underset{\substack{\uparrow \\ 2\times 2 \\ \text{minors}}}{I} \subset k[\underset{\substack{\uparrow \\ \text{entries of a} \\ 3\times 3 \text{ matrix.}}}{x_1, \dots x_9}] =: R$

$R/_{I}$ is not l.c.i

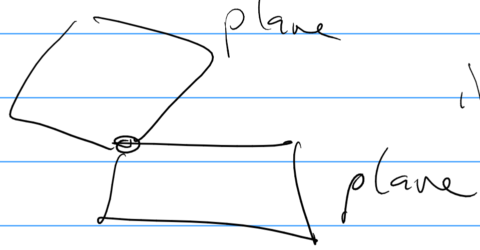
but it is CM.

(4) non-examples from last worksheet

$k[x,y,z]/_{(xy,xz)}$

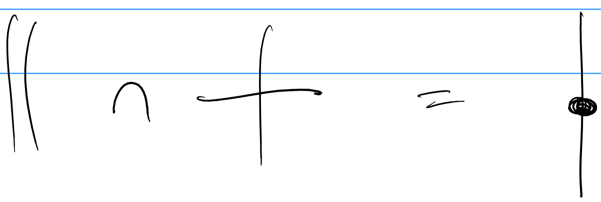


$k[x,y,z,w]/_{(x,y)\cap(z,w)}$



in k^4

$k[x,y]/_{x^2,xy}$



(Read Eisenbud §18.2 if you want)

Comment on last example:

Friday we said that

$$\dim R - 1 \leq \dim R/r \leq \dim R$$

and if r is not a zero-div then
$$\dim R/r = \dim R - 1$$

in $R = k[x,y]/x^2, xy$ loc at (x,y)

$\nexists$ non-zero-divisors.

nonetheless $\dim R = 1$
$$\dim R/y = \dim k[x]/x^2 = 0$$

OTOH $\dim R/x = \dim k[y] = 1$

more non-examples:

Eisenbud's fave is $k[s^4, s^3t, st^3, t^4] \subset k[s,t]$

i.e. the image of
$$k[x,y,z,w] \longrightarrow k[s,t]$$
$$x \longmapsto s^4$$
$$y \longmapsto s^3t$$
etc.

dim = 2 but depth $\underline{m}$ = 1.

Sm. curve of deg ≥ 3

my fave: $\mathbb{P}^1 \times C \subset \mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$

take affine cone over $\mathbb{P}^1 \times C$ in $\mathbb{A}^6$.

this is normal, unlike the others.