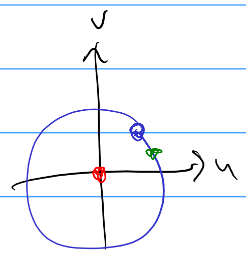
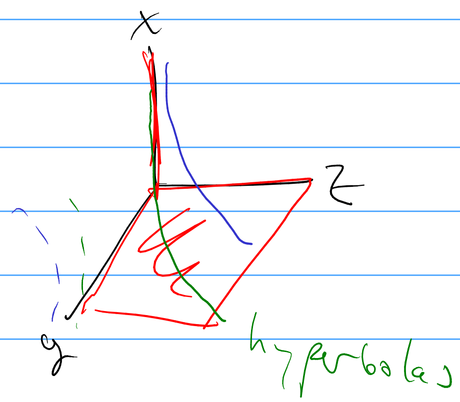
Last worksheet:

$$R = k[u,v] \longrightarrow S = k[x,y,z]$$
$$u \longmapsto xy$$
$$v \longmapsto xz$$

$$k^2 \xleftarrow{F} k^3$$
$$(xy, xz) \longleftarrow (x,y,z)$$





computed $\mathrm{Tor}_*^R\left(R/(u,v), S\right) = S/(xy,xz), S/x, 0 \ldots$

$\mathrm{Tor}_*^R\left(R/(u-1,v-1), S\right) = S/(xy-1, xz-1), 0, 0 \ldots$

Tor_1 is supported at the piece of $F^{-1}(0,0)$ that's bigger than expected.

$\mathrm{Tor}_*^R\left(R/u, S\right) = S/xy, 0, 0 \ldots$

$F^{-1}(v\text{-axis}) =$ two planes.

Today

R still a Noeth. local ring
$I \subset R$ proper ideal

seen: depth I $\leq$ codim / height $I \leq \dim R - \dim R/I$

def: R is Cohen-Macaulay if

$$\text{depth } \underline{m} = \dim R$$

[R Noeth. comm. but not local
is CM : iff $R_{\underline{m}}$ is CM $\forall$ max'l $\underline{m} \subset R$
iff $R_{\mathfrak{p}}$ is CM $\forall$ prime $\mathfrak{p} \subset R$
iff completion of $R_{\underline{m}}$ or $R_{\mathfrak{p}}$
is CM $\forall$ $\underline{m}$ or $\mathfrak{p}$

read Eisenbud Prop 18.8]

last time I mentioned:
take $X = \mathbb{P}^1 \times$ cubic curve $\subset \mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$
then affine cone $\hat{X} \subset \mathbb{A}^6$ is not CM.
reason: $H^1(\mathcal{O}_X) \neq 0$

more generally, given $X \subset \mathbb{P}^n$
the affine cone $\hat{X} \subset \mathbb{A}^{n+1}$ is CM
iff $H^i(\mathcal{O}_X(t)) = 0 \quad \forall t \in \mathbb{Z} \quad \forall\, 0 < i < \dim X$
and X is "projectively normal"

uses local cohomology (Eisenbud App. 4)

Thm if R is CM then every $I \subsetneq R$ has

$$\operatorname{depth} I = \dim R - \dim R/I$$

Pf following Eisenbud Thm 18.7

induct on $\dim R/I$.

if $\dim R/I = 0$ then

$\mathfrak{m}$ is the unique prime $\supset I$

so $\sqrt{I} = \bigcap$ all primes $\supset I$
$= \mathfrak{m}$

today's worksheet: $\operatorname{depth} I = \operatorname{depth} \sqrt{I}$

so $\operatorname{depth} I = \operatorname{depth} \mathfrak{m} = \dim R$ ✓

if $\dim R/I > 0$,

let $P_1, \dots, P_k$ be the minimal primes over I.

each $P_i \subsetneq \mathfrak{m}$ else R/I would be 0-dimensional

claim (prime avoidance):
$\exists\, r \in \mathfrak{m}$ $r \notin$ any P_i

[if R contains an infinite field, easy.]

now $\dim\left(R/_{I+(r)}\right) = \dim R/_I - 1$

[recall from last Friday:
take the longest chain of primes in $R/_{(I+(r))}$

lift to $R/_I$
Smallest $\mathfrak{p}$ in the chain is not minimal because it contains r,
so you can make it longer by 1.]

by induction,

$$\operatorname{depth}(I+(r)) = \dim R - \dim R/_{I+(r)}$$

$$= \dim R - \dim R/_I + 1$$

OTOH, $\operatorname{depth}(I+(r)) \leq \operatorname{depth}(I) + 1$

(using the s.e.s. of complexes

$$\operatorname{Kosz}(\text{gens } I) \to \operatorname{Kosz}(\text{gens } I, r) \to \operatorname{Kosz}(\text{gens } I)[1]$$

and look at the long exact seq in H_*)

so $\operatorname{depth} I \geq \operatorname{depth}(I+(r)) - 1$

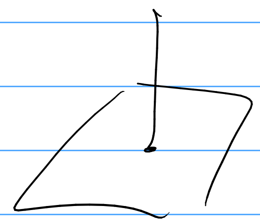
$$= \dim R - \dim R/_I + 1 - 1$$

$\blacksquare$

What about the claim (prime avoidance)?
Read Eisenbud Lemma 3.3 if you want.

examples: $I = (xy, xz) \subset R = k[x,y,z]_{(x,y,z)}$

minimal primes of I
are $p_1 = (x)$
$p_2 = (y,z)$

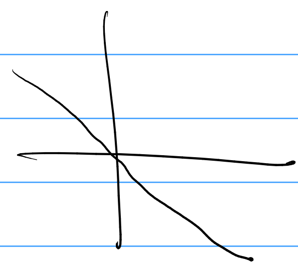


Want $r \in (x,y,z) \setminus (p_1 \cup p_2)$

$r = y$ avoids p_1
$r = x$ avoids p_2
$r = x+y$ avoids both.

② $I = (xy(x+y)) \subset R = k[x,y]_{(x,y)}$

$p_1 = (x)$
$p_2 = (y)$
$p_3 = (x+y)$



y avoids p_1
x avoids p_2
$x+y$ avoids p_1 and p_2 but not p_3 ..

Eisenbud's proof will produce $r = x+y(x+y)$