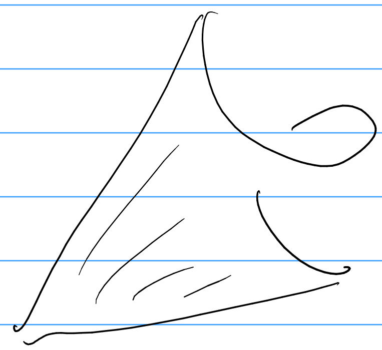
if $X \subset \mathbb{C}^n$ cut out by a radical ideal $(f_1, \dots, f_e) \subset \mathbb{C}[x_1 \dots x_n]$

goal: if X is smooth at a point $p \in \mathbb{C}^n$ then after localizing at corresp. max'l ideal, I is gen'd by a reg. seq of length $=$ codim X and R/I is regular.

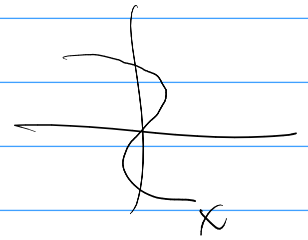
example: $I = (xz - y^2, xw - yz, yw - z^2) \subset \mathbb{C}[x, y, z, w]$

$X =$  $\subset \mathbb{C}^4$

X is smooth away from 0.

at 0, I really needs 3 gens.

but localize at any other point we can get away with 2 gens. and they form a reg. seq.



point $\rightsquigarrow$ $0 \in \mathbb{C}^n$

$R = \mathbb{C}[x_1 \cdots x_n]_{(x_1, \cdots x_n)}$

$I = (f_1, \ldots, f_e) \subset \underline{m} = (x_1, \ldots, x_n)$

suppose X is a manifold in an analytic nbd of 0

after permuting coords, can say

$x_1 \cdots x_d : X \longrightarrow \mathbb{C}^d$

is biholomorphic in some (analytic) nbd of 0.

let $S = R/I$, $\bar{m} \subset S$ max'l ideal.

I claim $x_1 \cdots x_d$ gen. $\bar{m}$ and they're a reg. seq.

i.e. the Koszul complex of $x_1 \cdots x_d$ in S has $H_0 = \mathbb{C}$ $H_{>0} = 0$ ✓

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consider R' = ring of germs at 0 of holo. functions on $\mathbb{C}^n$

Huybrechts cx geom. ch. 1

R'' = ring of formal power series $\mathbb{C}[[x_1, \cdots x_n]]$

$R \subset R' \subset R''$

$H^0(\mathrm{Kos}) = S/(x_1 \cdots x_d)$

$S/\bar{m} = \mathbb{C}$

claim: each is faithfully flat
over the previous one.
for R'', read Eisenbud Thm 7.2(6)
(proved in §7.5)

so $H_i(\text{Koszul}) \otimes R' = H_i(\text{Koszul} \otimes R')$

know that the map
$T := \mathbb{C}[x_1 \dots x_d]_{\text{loc at } 0} \longrightarrow S = R/I$

becomes an iso when you $\otimes R'$ or R''

and turns Koszul cx of $x_1 \dots x_d$ in T
(which we know all about)

into the Koszul — — — in S
which we wanted to study.

So R/I is a regular local ring.
in particular it's CM.

could now prove:
if R is a reg. loc. ring
and R/I is reg

then I is gen'd by a reg seq.

read the proof many places.
let's do something different.

still have

$I = (f_1, \dots, f_e) \subset \mathfrak{m} \subset R$

$\bar{\mathfrak{m}} \subset S = R/I$

notice: $\bar{\mathfrak{m}} = \mathfrak{m}/I$.

want to choose $f_1 \dots f_{n-d}$ that are a reg. seq and gen. I

have:

$$\begin{array}{ccccccccc} & & R^e & & & & & & \\ & & \downarrow{\scriptstyle (f_1, \dots, f_e)} & & & & & & \\ 0 & \to & I & \to & \mathfrak{m} & \to & \bar{\mathfrak{m}} & \to & 0 \end{array}$$

apply $- \otimes R/\mathfrak{m}$

$$\begin{tikzcd} \mathbb{C}^e \arrow[d] \arrow[dr] & & \\ I \otimes R/\mathfrak{m} \arrow[r] & \mathfrak{m} \otimes R/\mathfrak{m} \arrow[r] & \bar{\mathfrak{m}} \otimes R/\mathfrak{m} \arrow[r] & 0 \end{tikzcd}$$

$\mathfrak{m} \otimes R/\mathfrak{m} = \mathbb{C}^n$, $\quad \bar{\mathfrak{m}} \otimes R/\mathfrak{m} = \mathbb{C}^d$

this map is $J = \left(\dfrac{\partial f_j}{\partial x_i} \right)$ Jacobian matrix.

rank of J must be $n-d$.

permute $f_1, \dots, f_e$ so that the cols of J from $f_1, \dots, f_{n-d}$ span image of J.

notice: $f_1 \dots f_{n-d} : \mathbb{C}^n \longrightarrow \mathbb{C}^{n-d}$

is a submersion at $0 \in \mathbb{C}^n$
so fiber over $0 \in \mathbb{C}^{n-d}$ is a mfld near $0 \in \mathbb{C}^n$.

also: $x_1 \dots x_d \in \underline{m}$ map to gens of $\bar{m}$

so image in $\underline{m} \otimes R/\underline{m}$ is complementary subspace to im J.

if we look at

$$x_1, \dots, x_d, f_1, \dots, f_{n-d} : \mathbb{C}^n \longrightarrow \mathbb{C}^n$$

its jacobian is invertible at 0,
so local diffeo.

so gives an isomorphism
$$R' \longrightarrow R'$$

that turns I into $(x_{d+1}, \dots, x_n)$

thus $I \otimes R/\underline{m} = I \otimes R'/\underline{m} = \mathbb{C}^{n-d}$

the map $R^{n-d} \longrightarrow I$ become surjective after $\otimes R/\underline{m}$, so by Nakayama

it was already surjective.

so $f_1 \dots f_{n-d}$ generate I. $\boxtimes$

What have we learned?

- You first meet Ext and Tor over $\mathbb{Z}$.

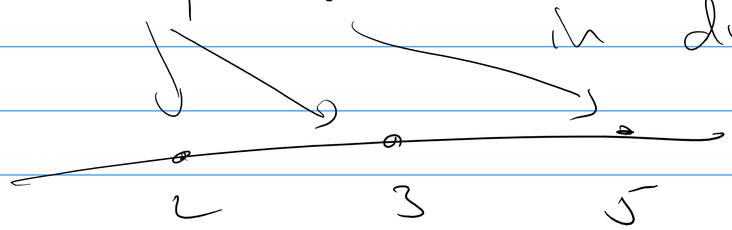
  but proj dim $\mathbb{Z} = 1$,

  like $k[x]$. - not too exciting.

  $k[x,y]$ $k[x,y,z]$ etc... gets more exciting.

- Think geometrically:

  $\mathrm{Tor}_1(\mathbb{Z}/6, \mathbb{Z}/5) = 0$ because they live in different places



  $\mathrm{Tor}_1(\mathbb{Z}/6, \mathbb{Z}/10) = \mathbb{Z}/2$ excess intersection at 2.

- Don't be afraid to localize.

- Over a Noeth. comm ring, a fin gen module is

  proj. $\Leftrightarrow$ flat $\Leftrightarrow$ loc. free.

proj. dim (M) measured by $\mathrm{Tor}_*(M, R/\underline{m})$

as $\underline{m}$ varies.

think about $\dim_{R/\underline{m}}(M \otimes R/\underline{m})$

as it varies from point to point.

loc const $\rightsquigarrow$ M is flat

jumps $\rightsquigarrow$ M not flat there.

- for Noeth local ring,

$$\operatorname{depth} \underline{m} \leq \dim R \leq \# \text{ gens } \underline{m}$$

gl.dim $R < \infty$ iff all 3 are $=$.

aka $\underline{m}$ generated by a reg. seq.

(R is a reg. loc. ring)

- many examples, lots of Koszul complexes.
- use Macaulay2 if it helps you.
- If you find yourself breaking a long exact seq into short exact seqs, should be using a spectral seq.

Thanks!