Worksheet 12

Math 607, Homological Algebra

Wednesday, April 29, 2020

1. Consider the ring $R = \mathbb{Z}[\sqrt{-3}]$ and the maximal ideal $\mathfrak{m} = (2, 1 + \sqrt{-3})$. Adapt your Macaulay2 code from last time to study $\mathrm{Tor}_*^R(R/\mathfrak{m}, R/\mathfrak{m})$.

   ```
   R = ZZ[y]/(y^2+3)
   m = ideal(2,1+y)
   C = resolution(R^1/m, LengthLimit => 6)
   ```

   Have a look at the differentials `C.dd_1`, `C.dd_2`, etc. They should remind you of the 2-periodic resolution from worksheet 7, but now instead of $xz + yw = 0$ you have

   $$2 \cdot 2 + (1 + \sqrt{-3})(-1 + \sqrt{-3}) = 0.$$

   Continue as you did last time:

   ```
   k = R/m
   C' = C ** k
   ```

   Have a look at the differentials of `C'`, and conclude that

   $$\mathrm{Tor}_i^R(R/\mathfrak{m}, R/\mathfrak{m}) = R/\mathfrak{m}$$

   for all i. So R has infinite global dimension, unlike $\mathbb{Z}[\sqrt{-5}]$ or $\mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$.

2. If you have extra time: We studied $\mathbb{Z}[\sqrt{-5}]$ and $\mathbb{Z}[\sqrt{-3}]$, which are flat extensions of $\mathbb{Z}$, and the self-Tors of their quotients by maximal ideals, finding that the first has global dimension 1, but the second has global dimension ∞ because of the maximal ideal $(2, 1 + \sqrt{-3})$. Do a similar analysis of $\mathbb{R}[x, y]/(y^2 - x^3 + x)$ and $\mathbb{R}[x, y]/(y^2 - x^3 - x^2)$, which are flat extensions of $\mathbb{R}[x]$. Or use $\mathbb{C}$ coefficients if you prefer.