Worksheet 17

Math 607, Homological Algebra

Wednesday, May 13, 2020

1. Let $R = k[x, y, z]/(x^2 + y^2 = z^2)$, and let $\mathfrak{m}$ be the maximal ideal (x, y, z). Geometrically, we're talking about the origin in a cone in 3-space.

   Ask Macaulay2 for the Koszul complex of the sequence x, y, z, and for (the beginning of) the minimal resolution of $R/\mathfrak{m}$:

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   R = QQ[x,y,z]/(x^2+y^2-z^2)
   K = koszul vars R
   C = resolution(R^1/(x,y,z))
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   Observe that for each i, the rank of the free module K_i is less than or equal to the rank of C_i. Squint at the differentials until you can say "it's plausible that K is a subcomplex of C."

2. Same with $R = k[x, y, z]/(xy, xz)$ and $\mathfrak{m} = (x, y, z)$, which you've studied before: geometrically, we're talking about the origin in the union of the x-axis and the yz-plane in 3-space.

3. If you have time, consider how to start proving that this happens in general. So let R be a Noetherian local ring, let $r_1, \dots, r_n$ be minimal generators for the maximal ideal $\mathfrak{m}$, and consider the first two steps of the Koszul complex of $r_1, \dots, r_n$ mapping to the first two steps of the minimal resolution of $R/\mathfrak{m}$:

$$\begin{tikzcd}
R^{\binom{n}{2}} \arrow[r] \arrow[d, "f"'] & R^n \arrow[r] \arrow[d, equal] & R \arrow[r] \arrow[d, equal] & R/\mathfrak{m} \arrow[r] \arrow[d, equal] & 0 \\
R^m \arrow[r] & R^n \arrow[r] & R \arrow[r] & R/\mathfrak{m} \arrow[r] & 0.
\end{tikzcd}$$

   Note that f exists because the bottom row is exact, even if the top row is not. The claim is that f is split injective; think about why.

   You can look at the proof of Eisenbud, Lemma 19.13, but you might get further thinking about it on your own.