Worksheet 22

Math 607, Homological Algebra

Friday, May 29, 2020

From today's lecture it follows that if R is a Noetherian local ring with maximal ideal $\mathfrak{m}$, then $\operatorname{depth}\mathfrak{m} \leq \dim R$.

Here are some examples where it is strictly less. We'll use maximal ideals $\mathfrak{m}$ in non-local rings S, but the numbers we'll get are the same as they would be for the maximal ideal in $S_{\mathfrak{m}}$.

1. Let $R = k[x, y, z]$, and let $I = (xy, xz)$ be the ideal that cuts out the union of the x-axis and the yz-plane in 3-space.

   Use Macaulay2 to compute the dimension of R/I:

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   R = QQ[x,y,z]
   I = ideal(x*y,x*z)
   S = R/I
   dim S
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   The answer is 2, which makes sense geometrically. Find a chain of prime ideals $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \mathfrak{p}_2 \subset R/I$.

   Next, recall from lecture 19 that the depth of $\mathfrak{m} = (x, y, z) \subset R/I$ equals d if and only if the homology of the Koszul complex vanishes in the last d places and no more. Use Macaulay2 to see which homologies vanish:

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   K = koszul vars S
   HH_0 K == 0 -- answer will be true or false
   HH_1 K == 0
   HH_2 K == 0
   HH_3 K == 0
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   Conclude that the depth of $\mathfrak{m}$ is 1, strictly less than 2. Find a non-zero-divisor in $\mathfrak{m}$.

2. Same with $R = k[x, y, z, w]$ and $I = (x, y) \cap (z, w) = (xz, yz, xw, yw)$, which cuts out two planes meeting at the origin in 4-space. The dimension of R/I is 2, but the depth of $\mathfrak{m} = (x, y, z, w) \subset R/I$ is 1.

3. Same with $R = k[x, y]$ and $I = (x^2, xy)$, which cuts out the y-axis together with a fat point at the origin in the plane. The dimension of R/I is 1, but the depth of $\mathfrak{m} = (x, y, z) \subset R/I$ is 0.