

Worksheet 23

Math 607, Homological Algebra

Monday, June 1, 2020

Let $R = k[u, v]$, let $S = k[x, y, z]$, and consider the ring map $\varphi: R \rightarrow S$ determined by $\varphi(u) = xy$ and $\varphi(v) = xz$. Geometrically, this corresponds to the map $F: k^3 \rightarrow k^2$ given by $F(x, y, z) = (xy, xz)$.

1. In k^3 , sketch $F^{-1}(u, v)$ for several points $(u, v) \in k^2$, including the origin.
2. You know that $R/(u, v)$ is resolved by the Koszul complex

$$0 \rightarrow R \xrightarrow{\begin{pmatrix} v \\ -u \end{pmatrix}} R^2 \xrightarrow{(u \ v)} R \rightarrow R/(u, v) \rightarrow 0,$$

so $\text{Tor}_*^R(R/(u, v), S)$ is computed by applying φ to that complex and taking homology. Find the answer in Macaulay2:

```
S = QQ[x,y,z]
K = koszul matrix{{x*y,x*z}}
prune HH_0 K
prune HH_1 K
prune HH_2 K
```

Notice that this is pretty much the same calculation you would do to compute the depth of the ideal (xy, xz) . Also notice that the support of Tor_1 is the yz -plane, and doesn't include the x -axis.

If you're in the mood, also look at $\text{Tor}_*^R(R/(u-1, v-1), S)$, which has to do with the fiber $F^{-1}(1, 1)$:

```
K = koszul matrix{{x*y-1,x*z-1}}
prune HH_0 K
prune HH_1 K
prune HH_2 K
```

3. Look at $\text{Tor}_*^R(R/u, S)$, which has to do with F^{-1} of the line $u = 0$:

```
K = koszul matrix{{x*y}}
prune HH_0 K
prune HH_1 K
```