One asked: is exact equal to presence inj. and surj?

No: let \( R = k \) so \( R - \text{mod} = k - \text{vec} \)

Functor: \( \Lambda^n \) (exterior power)

if \( 0 \to U \to V \) then \( 0 \to \Lambda^n U \to \Lambda^n V \)
(fundamental)

if \( V \to W \to 0 \) then \( \Lambda^n V \to \Lambda^n W \to 0 \)

but if \( 0 \to U \to W \to 0 \)
then \( 0 \to \Lambda^n U \to \Lambda^n W \to 0 \)

is almost never exact.

\[
0 \to C^2 \to C^1 \to C^2 \to 0
\]

apply \( \Lambda^2 \)

\[
0 \to C^1 \to C^6 \to C^1 \to 0
\]

can’t be exact.

Further Q: does additive save us?
So far this week: R comm. Noetherian
M a fin. gen. module.

\[ M \text{ flat } \iff M \otimes R \text{ flat} \]
\[ M \text{ proj } \iff M \otimes R \text{ proj} \]

Many examples + digressions.

Today: R Noetherian local ring

= comm w/ unique max'l m \subset R

\[ k := R/m \quad \text{residue field} \]

For example:

\[ R = \mathbb{Z}/6 \text{ odd} \quad m = (2) \quad k = \mathbb{F}_2 \]

\[ R = k[x,y]/(x^2, y^2) = \left\{ \frac{f}{g} \mid g(0,0) \neq 0 \right\} \quad m = (x,y) \quad k = k \]

One more:

\[ R = \text{power series in } x,y \text{ coeffs in } k \]

with \( r \) radius of conv.

(If we said radius of conv \( r \) fixed \( r \)
not local: max'l ideal \( A \) point in open disc of radius \( r \).
But if we don't bound radius (below...) \( r \))
Nakayama's lemma: \( R, m, k \) as above
\( M \) a fin. gen. module.

Then \( M = 0 \) if \( M \otimes_k k = 0 \).

Proof: notice \( M \otimes_k k = M/\mathbb{m}M \).

(why? take \( 0 \to M \to R \to k \to 0 \))

and \( \mathfrak{m} \) in \( R \).

so \( M \otimes_k k = 0 \) if \( M = \mathfrak{m}M \).

let \( u_1, \ldots, u_r \in M \) be generators.

each \( u_i \in \mathfrak{m}M \), so write
\[ u_i = \sum a_{ij} u_j \quad a_{ij} \in \mathfrak{m} \]

Consider
\[ A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots \\ a_{21} & a_{22} & a_{23} & \cdots \\ a_{31} & a_{32} & a_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \]

then \( A \cdot \begin{pmatrix} u_1 \\ \vdots \\ u_r \\ \mathbf{0} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \vdots \\ \mathbf{0} \end{pmatrix} \)

is \( A \) invertible? then we wish \( u_i = 0 \) so \( M = 0 \).
\[ A = -I \mod m \]
\[ \det A = \pm 1 \mod m \]
so \( \det A \not\equiv m \)
so \( \det A \) is a unit (since \( R \) is local!)
so \( A \) is invertible. \( \blacksquare \)

Counterexample when \( M \) is not fin gen.
\[ R = \mathbb{F}_2, \quad M = \mathbb{Q}, \quad M \otimes_k \mathbb{Z}/2 = 0 \]
Similar: \( R = k(x,y), \quad M = k(x,y), \quad \text{field of rational functions} \)
\[ L = \mathbb{R}(x,y), \quad M \otimes_k L = 0. \]

Worksheet: if \( N \) is fin gen
then \( f: M \rightarrow N \) is surj.
iff \( f_0: M \otimes_k N \otimes_k \) is surj.
(Doesn't work with injective.)
Corollary: Let $M$ be fin-gen then minimal # of generators $= \dim_k (M \otimes k)$. 

Proof: Choose basis $v_1 \ldots v_r$ for $M \otimes k = M / \mathfrak{m}$. Lift these to $u_1 \ldots u_r \in M$. Then they generate $M$. 

$$\begin{align*}
\mathbb{R}^r &\rightarrow M \rightarrow \mathfrak{m} 
\text{ is surj}, \\
\text{iff } \mathbb{R}^r &\rightarrow M \otimes k \rightarrow \mathfrak{m} 
\text{ is surj}.
\end{align*}$$

Then $\mathbb{R}$ Noetherian local ring $M$ a fin-gen module. 

Free $\Rightarrow$ proj $\Rightarrow$ flat. 

Proof in general, free $\Rightarrow$ proj $\Rightarrow$ flat, so suppose $M$ is flat. 

Choose minimal generators 

$$\begin{align*}
\mathbb{R}^r &\rightarrow M \rightarrow \mathfrak{m}
\end{align*}$$
$0 \to \ker \to R^5 \to M \to 0$

$\otimes_k$

$\text{Tor}^R(M,k) \to \ker \otimes_k \mathbb{L} \to \mathbb{L} \to \text{Mat} \to 0$

$\sim$

Zero because $M$ is flat.

So $\ker \otimes_k \mathbb{L} = 0$.

$\ker$ is fin. gen. because $R$ is Noeth.

So $\ker = 0$

So $M \cong R^5$.

$\therefore$

Next week: Ext and Tor