On Worksheet 3, had

\[ R = k[x,y]/(x^3 - x - y^2) \]

look at \( k = R/(x,y) \)

\[ x^3 - x - y^2 = x(x^2 - 1) + y(-y) \]

relate to Friday's discussion of \( x^2 + y^2 \)

free res.: \( (y, x^2 - 1) \)

\[ 0 < k < R \xrightarrow{\psi} R^2 \xrightarrow{k \cdot} R \xrightarrow{\psi} R^2 \xrightarrow{k \cdot} \text{repeat} \]

still exact - adapt argument from Friday.

looks like proj. dim \( (k) \) might be 2?

compute \( \text{Tor}_k^R(k,k) \):

take that free res. and \( \otimes k = R/(x,y) \)

\[ k \xrightarrow{\otimes k} k \xrightarrow{\otimes k} k \xrightarrow{(0,0)} k \xrightarrow{\otimes k} \text{repeats.} \]

\[ \text{Tor}_0 = k \]

\[ \text{Tor}_1 = k/k^1 = k \]

\[ \text{Tor}_2 = k^1/k^1 = 0 \]

\[ \text{Tor}_3 = k^1/k^1 = 0 \quad \text{etc.} \]
Maybe proj. dim \( R(k) = 1 \), not \( \infty \)?

Truncate res. above

\[ 0 \to k \to R \to \text{ideal}(x,y) \to 0 \]

\( \to \) saw that this was loc. free.

Similar: \( R = \mathbb{Z}[\sqrt{-5}] \), \( I = (2, 1 + \sqrt{-5}) \) is loc. free

\[ 0 \to \mathbb{Z}[\sqrt{-5}] \to \mathbb{Z}[\sqrt{-5}] \to \mathbb{Z}[\sqrt{-5}]^2 \to \to 0 \]

\[ 0 \to R \to R \to R \to \to 0 \]

Was it a coincidence that computing \( \text{Tor}_2^R(k,k) \) told us proj. dim \( R(k) \)?

No: can detect proj. dim \( (k) \)

using \( \text{Tor}_2^R(M, R(m)) \) as \( m \in R \) max.\'t ideal varies.

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Now: Ext and Tor are well-defined.
Recall: a (chain) complex of R-modules is

\[ \cdots \rightarrow M_i \xrightarrow{d_i} M_{i-1} \xrightarrow{d_{i-1}} M_{i-2} \xrightarrow{d_{i-2}} \cdots \]

\[ d^2 = 0, \quad \text{im } d_{i+1} \subset \ker d_i \]

**Homology:** \( H_i(M) = \ker d_i / \text{im } d_{i+1} \)

A chain map \( f : M \rightarrow N \)

\[ \cdots \rightarrow M_i \xrightarrow{d_i} M_{i-1} \xrightarrow{d_{i-1}} M_{i-2} \xrightarrow{d_{i-2}} \cdots \]

\[ \xrightarrow{f} \]

\[ \cdots \rightarrow N_i \xrightarrow{d'_i} N_{i-1} \xrightarrow{d'_{i-1}} N_{i-2} \xrightarrow{d'_{i-2}} \cdots \]

\[ fd = df \]

\( f \) takes \( \ker d_i \) into \( \ker d'_i \),
\( \text{im } d_{i+1} \) into \( \text{im } d'_{i+1} \),

so induces a map \( H_i(M) \rightarrow H_i(N) \)

Two chain maps \( f, g \) are **homotopic** if \( s : M_i \rightarrow N_{i+1} \) s.t. \( f - g = ds + sd \)

(Note that \( s \) does not commute with \( d \).)

Then \( f \) and \( g \) induce the same map \( H_i(M) \rightarrow H_i(N) \).
Favorite example: \( X \) fin. Space
\[
C_i(X; \mathbb{R}) = \mathbb{R}
\]
\[\oplus \text{ cont maps } \sigma : \Delta^i \to X \to \mathbb{R}\]
\[d : C_i \to C_{i-1} \text{ takes the boundary } \]

given \( f : X \to Y \) cont., get a chain map
\[
f : C_\ast(X; \mathbb{R}) \longrightarrow C_\ast(Y; \mathbb{R})
\]
\[\sigma : \Delta^i \to X \to f \circ \sigma : \Delta^i \to Y \]

given \( f, g : X \to Y \) and a homotopy \( h : X \times I \to Y \),
build a chain map \( s : C_\ast(X; \mathbb{R}) \to C_{\ast+1}(Y; \mathbb{R}) \)
between \( f_\ast \) and \( g_\ast \),

Also recall the univ. prop. of proj. modules:
\[
\begin{array}{ccc}
    & X & \\
\sigma & \downarrow & \phi \\
    & \leftarrow & N \\
\end{array}
\]
\[M \longrightarrow N \longrightarrow 0\]
\( \text{Tor}_i^R(M, D) \) is functorial in \( M \) or \( D \)

\( \Rightarrow \) easy work

Given \( f: M \rightarrow N \), want \( f_*: \text{Tor}_i^R(M, D) \rightarrow \text{Tor}_i^R(N, D) \)

\[
\begin{array}{c}
\cdots \rightarrow P & \xrightarrow{d_2} & P & \xrightarrow{d_1} & P_0 & \xrightarrow{\epsilon} & M \rightarrow 0 \\
\cdots \rightarrow P' & \xrightarrow{d_2'} & P' & \xrightarrow{d_1'} & P'_0 & \xrightarrow{\epsilon'} & N \rightarrow 0
\end{array}
\]

Observe that \( \epsilon' f_0 d_1 = f \in d_1^* d_0 = 0 \),

so \( f_0 d_1 \) takes values in \( \ker \epsilon' = \text{im} d_1' \),

and \( P' \xrightarrow{d_1'} \text{im} d_1' \subset P'_0 \)

This induces a map on \( \text{Tor}_i^R \).

Apply -\( \otimes D \)

Still have two chain complexes

and a chain map

we get induced map on \( H_i = \text{Tor}_i^R(-, D) \)
Claim: given two lifts $f_i, f_i'$ of $f$, they induce the same map on $\text{Tori}$.

reduce to: $f_i - f_i'$ which lifts $\sigma : M \to N$, reduces $0'$ on $\text{Tori}$.

Worksheet: produce $s_i : P_i \to P_{i+1}$

with $ds + sd = f_i - f_i'$

Then we win.