on last worksheet,

\[ R = \frac{k[x,y,z]}{(xy, xz)} \]

\[ m_1 = (x-1, y, z) \quad \text{Tor}_1(R/m_1, R/m_1) = 1, 0, \ldots \]

\[ m_2 = (x, y-1, z) \]

\[ x = 0 \quad \text{Tor}_2 = 1, 2, 1, 0, \ldots \]

\[ m_3 = (x, y, z) \]

\[ \text{Tor}_3 = 1, 3, 5, 8, 13, 21, \ldots \quad (\text{Fibonacci!}) \]

\[ R_{m_1} \] is the same as \( k[x]/(x-1) \)

because in \( R_{m_1} \), \( x \) is a unit

\[ \frac{k[x,y,z]}{(xy, xz)} \cong \frac{k[x,y,z]}{(y,z)} \]

Could have computed

\[ \text{Tor}_1(R_k, k(x) \frac{k(x)}{x-1}) = 1, 1, 0, 0, \ldots \]

easily using \( \phi : k(x) \xrightarrow{x-1} k(x) \rightarrow k(x)/x-1 \).
\[ \mathfrak{m}_y = \kappa [y, z](y - t) \]

Because \( y \) is a unit in \( R_{\mathfrak{m}_y} \)

\[ k[x, y, z]/(ky, x) \rightarrow k[x, y, z]/(ky, x) \]

Could have computed

\[ \text{Tor}_1(k[y, z], \frac{k[y, z]}{(y, x)}) = 1, 1, 0, 0, ... \]

easily with a 2-step Koszul complex that we've studied.

\( R_{\mathfrak{m}_y} \) is really as complicated as it looks in Macaulay 2.

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Corey points out: that isn't a field.

Every \( P(0) \) has glob dim 1. Why? Let \( m \in R \) be maximal.

Write \( m = (r) \) \( r \neq 0 \)

Then \( 0 \rightarrow R \rightarrow R \rightarrow R/m \rightarrow 0 \)

So proj. dim \( (R/m) = 1 \)
so \( k[x, y, z]/(z^2 - 5) \)
all have glob. dim 1.

but \( k[x, y]/y^2 - 5 \) is not a PID and still has glob. dim 1.

Details

\[ k \rightarrow R = k[x, y]/y^2 - 5 \]

geometric analogy:

\[
\begin{align*}
\mathbb{R}[x] & \rightarrow \mathbb{R}[x, y]/y^2 - x^3 - x \\
\mathbb{C}[x] & \rightarrow \mathbb{C}[x, y]/y^2 - x^3 - x
\end{align*}
\]

Claim is that \( \operatorname{Tor}_{2n}^R (R/I_1, R/I_2) = 0 \)

if \( I_1 \subseteq I_2 \subseteq R \).

what do \( m \)'s look like?

Consider \( m = \mathfrak{m} = p^k \) for some prime \( p \).

\( pR \subset m \)
Compute \( \mathbb{Z}/(\mathbb{Z}/p^i \mathbb{Z}) = \begin{cases} \mathbb{Z}/p & i = 0 \\ \mathbb{Z}/p^i & i = 1 \\ \mathbb{Z}/p^{i-2} & i \geq 2 \end{cases} \)

(use 0 \to \mathbb{Z} \to R \to R/p \to 0)

saw that \( R = \mathbb{F}[y]/(y^2 + 5) \)

so \( R/p = \mathbb{F}_p[y]/(y^2 + 5) \)

Case 1: if \( y^2 + 5 \) is irreducible

then \( R/p \) is a field \( (\mathbb{F}_p^2) \)

so \( pR \) is maximal

so \( pR = m \).

Case 2: if \( y^2 + 5 = (y - a)(y - b) \) in \( \mathbb{F}_p[y] \)

then \( R/p = \mathbb{F}[y]/(y - a)(y - b) \)

\[ = \mathbb{F}[y]/y-a \otimes \mathbb{F}[y]/y-b = \mathbb{F} \oplus \mathbb{F} \]

maximal ideals of \( R \) over \( pR \)

\( \leftrightarrow \) maximal ideals of \( R/p \)

two of them \( \leftrightarrow \) \( y-a \) \( y-b \)
one \( \cong m \), call the other \( m' \):

\[
R/p = R/m \oplus R/m'
\]

\[
\text{Tor}_1(R/p, R/p) = \text{Tor}(R/m, R/m') \oplus \text{Tor}(R/m', R/m')
\]

This vanishes above deg 1, so there do too.

Case 3: \( y^2 + 5 = (y-a)^2 \) in \( \mathbb{F}_p[y] \)

Think... either \( p = 2 \) or \( p = 5 \)

if \( p = 5 \), \( R/p = \mathbb{F}_5[y]/y^2 + 5 \)

maximal ideals over \( pR \)

\( \rightarrow \) max. ideals over 0 in there

just \( (y) \)

\( m = (\sqrt{5}) \) principal \( \Rightarrow R/m \) proj. dim 1.

\( 0 \rightarrow R \rightarrow R^2 \rightarrow R/m \rightarrow 0 \)
If $p = 2$ then $R/p = \mathbb{F}_2[y]/y^2 + y = \mathbb{F}_2[y]/(y+1)$

maximal ideals over $\mathbb{Z}$ is $(y+1)$

so $\mathfrak{m} = (2, 1 + y - y^2)$

not a principal ideal, but it's proj. as an $R$-module

$0 \rightarrow \mathfrak{m} \rightarrow R \rightarrow R/\mathfrak{m} \rightarrow 0$

so proj. dim $R/\mathfrak{m} = 1$. 

Worksheet: $\mathbb{Z}/(5,3)$

glean analogue: $R[x] \rightarrow R[x,y]/(y^2 - x^3 + x \cdot 2)$