Last time: \( R = k(x_1, \ldots, x_n) \)

\( M = \mathbb{R}^n \) with basis \( \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \)

Euler 0.f. \( \xi \in M \\nM^* = \mathbb{R}^n \) with basis \( dx_1, \ldots, dx_n \)

\[ \begin{align*}
\text{Closed complex} \\
0 \rightarrow \Lambda^0 M^* \rightarrow \cdots \\
\Lambda^2 M^* \rightarrow M^* \rightarrow \mathbb{R} \rightarrow 0 \\
\uparrow \\
\text{homology here is } R(x_1, \ldots, x_n)
\end{align*} \]

\( \text{but exact elsewhere.} \)

\( \text{if } \omega = f \, dx_1 \wedge \cdots \wedge dx_n \in \Lambda^k M^* \)

where \( f \) is homog. of deg \( c_i \)

\( \text{then } d(\mathcal{L}f \omega) + \mathcal{L}f \omega = (k+c) \omega \)

[Cartan's magic fork: it's \( \mathcal{L}f \omega \)]

\( \mathcal{L}f = \frac{\partial}{\partial t} \frac{\partial f}{\partial x} \)

\( \mathcal{L} \frac{\partial}{\partial x} = dx_i \)

\( \mathcal{L} \frac{\partial}{\partial x} \text{ obeys a Leibniz rule.} \)}
Let \( \Lambda^k \mathcal{M}^+ \) be general, \( k > 1 \).

Write \( \omega \) as a sum of Young pieces:

\[
\omega = \omega_0 + \omega_1 + \ldots + \omega_k
\]

if \( \sum \omega_i = 0 \) then each \( \sum \omega_i = 0 \)

so \( \sum \omega_i = (k+1) \omega_k \)

so \( \sum \left( \frac{1}{k} \omega_0 + \frac{1}{k+1} \omega_1 + \ldots + \frac{1}{k+2} \omega_k \right) = \omega \).

so the complex is exact where claimed.

(this proof requires that \( k = 0 \), but the conclusion is true more generally.)


if \( R = \{(x, \ldots, x_n) \} \) and \( m = (x_1, \ldots, x_n) \)

then \( \omega^R (R/m, R/m) = 1, 1, \ldots, 1, 0 \) —

resolve by Koszul cx.

\( R/m \) is all differentials \( = 0 \)

proj. dim \( R/m = n \).
If \( m = (x_1-a_1, \ldots, x_n-a_n) \)

then it's similar: proj dim \( R/m = n \).

If \( k = k \) that proves that

\[ \text{glob dim } R = n. \]

An answer to question:

\[ R = k[x_1, x_2, x_3, x_4], I = (x_2-y^2, x_3-yz, yw-z^2) \]

Koszul ex of \( k[x_1] \) is not exact.

\[ \text{if } k \neq k, \text{ still glob dim } k[x_1, \ldots, x_n] = n \]

know glob dim = n

because of \( \text{Tor} \) \( R/(x_1, \ldots, x_n), \text{self} \).

Let's prove \( \leq \).

\[ R = k[x_1, \ldots, x_n], S = k[x_1, \ldots, x_n] \cong R \otimes k \]

Claim: \( S \) is flat over \( R \)

Better: a seq of \( R \)-modules

\[ 0 \to L \to M \to N \to 0 \]

is exact

iff

\[ 0 \to R \otimes M \to R \otimes N \to 0 \]

is exact

"faithfully flat"
Pf  $\xrightarrow{0 \to L \to M \to N}$ is exact as $R$-mod
iff it's exact as $k$-vector spaces.
iff $0 \to \mathbb{E}_R \xrightarrow{M \otimes \mathbb{E}_k} N \otimes \mathbb{E}_k \to 0$

is exact as $\mathbb{E}_k$-vector spaces.
(think about it)

iff $0 \to \mathbb{L} \otimes S \xrightarrow{\mathbb{M} \otimes S} \mathbb{N} \otimes S \to 0$

is exact as $S$-modules.

Proof if $R \otimes S$ is faithfully flat
then an $R$-module $A$ is flat iff
$\Rightarrow A \otimes S$ is flat/\$S$.

PF is easy. For $\exists$

let $0 \to L \to M \to N \to 0$ be

an exact seq. of $R$-mods.

is $0 \to \mathbb{L} \otimes \mathbb{A} \to \mathbb{M} \otimes \mathbb{A} \to \mathbb{N} \otimes \mathbb{A} \to 0$ exact?

Yes iff same $\mathbb{L} \otimes \mathbb{S}$ is exact, but

$\mathbb{L} \otimes \mathbb{S} = \mathbb{L} \otimes (\mathbb{A} \otimes \mathbb{S})$ and both those

operations are exact by hypothesis. $\exists$
Prop: Let $M$ be a fin gen module over $R = k[x_1, \ldots, x_n]/J$. Then \( \text{proj dim } M \leq n \).

**Proof:** Take any fin gen proj res.

\[ \cdots \to P \to P_0 \to M \to 0 \]

Cut it off: let $K = \ker d_{n-1} = 0 \to K \to P \to \cdots \to P_1 \to P_0 \to M \to 0$.

Claim: $K$ is proj.

Know $K \otimes_R S$ is proj, because

$$\text{Tor}_i^S(K \otimes_R S, S/J_m) = \text{Tor}_i^S(M \otimes_R S, S/J_m) = 0$$

so $K \otimes_R S$ is flat.

so $K$ is flat.

so $K$ is proj. (bec. fin gen).

\qed