Working forward:
understand how Tor, Ext proj, dim interact with geometric idea of dim.

Warning: my indexing doesn't match Eisenbud's

\( R = \text{North local ring} \)

\( r_1, \ldots, r_n \in \mathbb{R} \)

\( K = \text{Koszul complex} \)

\[ 0 \to R \to R^n \to \cdots \to R^2 \to R \to \mathbb{R} \to 0 \]

this cone:
\( H_1 \to H_2 \to H_1 \to H_0 \)

Eisenbud:
\( H^0 \to H^1 \to \cdots \to H^n \)

my research this:
\( H^1 \to H^2 \to H^1 \to H^0 \)

last time:
\( r_1, \ldots, r_n \) is a reg. seq.

iff \( H_{\geq 1}(k) = 0 \)

iff \( H_1(k) = 0 \)
Thus: let \( k \) be the greatest int. s.t.
\[
th_k(k) \neq 0 \quad \text{but} \quad th_{k+1} = 0
\]
then any maximal reg seq. in
\[
i := (r, \ldots, r_n)
\]
has length \( n-k \).

\[ \text{Def: this is called the depth of } i \]

\[ \text{wants to be called } I \]

\[ \text{Lemma: every } r_i \text{ acts on every } th_j(k) \text{ by zero. } \]
\[ \text{hence every element of } I \text{ does. } \]

\[ \text{believable: } th_0(k) = R/I \]

\[ \text{claim is that the rest are } R/I \text{-modules. } \]

\[ \text{PF of Lemma: } \]
\[ r_i \ldots r_n \text{ give a map } \]
\[ R[x_1, \ldots, x_n] \xrightarrow{\phi} R \]
\[ x_i \mapsto r_i \]

\[ \text{kosz. complex of } x_1 \ldots x_n \in R(k) \text{ is exact. } \]
apply \( \Phi \) get \( k \).
Thus $H_i(k)$ computes $Tor_{i-1}^{R_{k+1}}(R^{(k)}, R)$.

Could also have resolved $R$ as an $R/I$-mod and applied $\otimes R/I$.

Which would kill all $F(x) = 0$.

If you know $H_k(k) \neq 0$, $H_{>k} = 0$

Let $y_1, \ldots, y_s \in I$ be a maximal reg seq.

Consider $Kosz (r_1, \ldots, r_n, y_i) =: K_i$.

$K \rightarrow K_i \rightarrow k[1]$ exact seq. of complexes

$\rightarrow H_i(k) \rightarrow H_i(k_i) \rightarrow H_{i-1}(k)$

$y_i$ act by 0

So $H_{k+1}(k_1) \neq 0$ and $H_{>k+1} = 0$

for $Kosz (r_1, \ldots, r_n, y_i, y_2)$

$H_{k+2} \neq 0$ and $H_{>k+2} = 0$

for $Kosz (r_1, \ldots, r_n, y_1, \ldots, y_s)$

$H_{k+s} \neq 0$ and $H_{>k+s} = 0$
other way:

\[ \text{Kosz}(y_1, \ldots, y_s) \text{ has } H_0 \neq 0 \quad H_{>0} = 0 \]

because \( y_1, \ldots, y_s \) is a reg. seq.

want to say:

\[ \text{Kosz}(y_1, \ldots, y_s, r) \text{ has } H_1 \neq 0 \quad H_{>1} = 0 \]

if \( H > 0 \) then \( y_1, \ldots, y_s, r \) is regular

so \( y_1, \ldots, y_s \) was not a maximal reg. seq.

what is \( H_1 \)?

\[
\begin{align*}
H_1(k(y)) & \to H_1(k(y, r)) \to H_0(k(y)) \to 0 \\
\to H_0(k(y)) & \to H_0(k(y, r)) \to 0
\end{align*}
\]

let \( S = \mathbb{R}/y_1, \ldots, y_s \). Then

\[ H_1(k(y, r)) = \left\{ z \in S \mid r, z = 0 \right\} \]

plausible:

in \( \text{Kosz}(y_1, \ldots, y_s, r, \ldots, r_n) \)

\[ H_0 = \left\{ z \in S \mid r, z = z_2 = \ldots = z_n = 0 \right\} \quad H_{>0} = 0 \]
assume for now that if $g_i, g_j$ was maximal then that $H_m = 0$ so $H_m (\text{last } g_i, g_j) = 0$ and $H_{>m} = 0$.

Saw above that $H_{<s} = 0$ and $H_{>k+s} = 0$.

Conclude that $k+s = n$ so $s = n-k$.

We've reduced to this claim:

$$\overline{g_1}, \ldots, \overline{g_s} \in S = R/\langle g_1, \ldots, g_s \rangle$$

all zero-divisors

If the map $S \xrightarrow{(i_1)} S^n$ were injective then some $S$-lin. combo of $\overline{g_i}$'s would be not a zero-div.

have to say "prime avoidance"

or "associated prime" unfortunately.

so just take it on faith. 

\[\text{NB: without the black box we proved} \quad S \leq n-k.\]