Where are these spectral seq coming from?

Def a double complex $C^{p,q}$ of $R$-modules has a horizontal diff $d_{\text{horz}}: C^{p,q} \to C^{p,q+1}$ and a vertical diff $d_{\text{vert}}: C^{p,q} \to C^{p+1,q}$.

They commute, and both have $d^2 = 0$.

The total complex $T = \text{tot}(C^{p,q})$ is given by $T^k = \bigoplus_{p+q = k} C^{p,q}$ and $d = d_{\text{vert}} + (\pm)^k d_{\text{horz}}$.

It satisfies $d^2 = 0$.

From a double complex, get a spectral seq $E^{p,q}_0 = C^{p,q}$, $d_0 = d_{\text{vert}}$.

\[\begin{array}{ccc}
0 & \to & C^{0,1} \\
\uparrow & & \uparrow \\
E^{0,1}_1 & \to & C^{1,1} \\
\uparrow & & \uparrow \\
C^{0,2} & \to & C^{2,0} \\
\end{array}\]
$E_i = \underline{H}^i_{\text{vert}} (C^{p,\ast})$

Think of $C^{p,\ast}$, $C^{1,\ast}$, $C^{2,\ast}$ as complexes
draw as maps of complexes
so $d_{\text{vert}}$ induces maps on $H^i$

still have $d^2 = 0$

$\mathbb{C}$ call this the complex $H_{\text{vert}}$

$E^0_i = \underline{H}^i_{\text{vert}} + \underline{H}^i_{\text{horz}}$

What does $d_2$ look like?

Given some $\alpha \in \underline{H}^0_{\text{horz}} (H_{\text{vert}})$,

lift to $\bar{\alpha} \in \underline{H}^1 (C^{0,\ast})$

Know that $d_1^{0,\ast} (\bar{\alpha}) = 0 \in \underline{H}^1 (C^{1,\ast})$

lift further to $\bar{\alpha} \in C^{1,\ast}$
so \( d_{\text{horz}} (x^2) \in C^{1,1} \)

and it's zero in \( H^1 (C^{1,1}) \)

so \( \exists \beta \in C^{1,0} \) such that

\[
d_{\text{vert}} \beta = d_{\text{horz}} x^2
\]

now define \( d_{\text{av}}^0 (x) = \left[d_{\text{horz}} \beta \right] \in H^2_{\text{horz}} (H^0_{\text{vert}}) \)

check: it's well-defined.

\[
(d_{\text{av}})^2 = 0
\]

Thus if \( C^{p,0} \) lives in the first or third quadrant then

spec. seq. \( \Rightarrow H^{p+q} \) (total complex).

rule: can also transpose \( C^{p,0} \)

\[
\text{tot} (C^{p,0}) = \text{tot} (C^{0,p})
\]

but the spectral sequence of the transpose gives different info.
Application: change of ring s.s. that we've been using

\[ R \rightarrow S \]
\[ m \text{ on } R \text{-mod} \]
\[ N \text{ on } S \text{-mod} \]

\[ E^{p, q}_{2} = \text{Tor}_{-p}^{S}(\text{Tor}_{p}^{R}(M, S), N) \Rightarrow \text{Tor}_{-p}^{R}(M, N) \]

To set it up, first resolve \( S \) by free \( R \)-modules

\[ \cdots \rightarrow R^{m-2} \rightarrow R \rightarrow R^{m} \rightarrow \cdots \rightarrow S \rightarrow 0 \]

Then resolve \( N \) by free \( S \)-modules

\[ \cdots \rightarrow S^{n-2} \rightarrow S^{n} \rightarrow \cdots \rightarrow N \rightarrow 0 \]

\[ \cdots \rightarrow R^{m, n-2} \rightarrow R^{m, n} \rightarrow \cdots \rightarrow R^{m, 0} \rightarrow \cdots \rightarrow R^{m, 0} \}

This is my double complex

\[ C^{p,q}_{i} = R^{m-n+p} \text{ with these differentials} \]

First claim: \( T := \text{tot} (C^{p,q}) \) is a res of \( N \) by free \( R \)-modules.
just run the s.s. on the double cx.

$E^0$ is above.

$E_{1}^{p,q}$ is $S^m \rightarrow S^n \rightarrow S^n$.

$E_{2}^{p,q}$ is $\cdots \cdots 0$

\[\begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{array}\]

$E_{3}^{p,q}$ is $\cdots \cdots 0$

$E_{\infty}^{p,q}$ is $\cdots \cdots 0$

so $\mathcal{T}$ is a complex of free $R$-mods with $H^0 = N$, otherwise exact.

now consider the double complex $M \otimes C_{i}^{q,p}$

$E_{0}^{q,p}$ is $M \otimes R^{m \times p} = M^{m \times p}$

$E_{1}^{q,p}$ is $\text{Tor}_{q}^{R}(M, S)^{m \times p}$

rows are $\text{Tor}_{q}^{R}(M, S)$ \otimes \text{free res of } N$

So $E_{2}^{p,q} = H^0_{m \times p} \text{Tor}_{q}^{R}(M, S)^{m \times p} = \text{Tor}_{q}^{R}(\text{Tor}_{p}^{R}(M, S), N)$

end of the s.s. is $H_{\infty}^{p,q}(M \otimes \mathcal{T})$

which is $\text{Tor}_{p}^{R}(M, N)$

because $\mathcal{T}$ was a res of $N$ by free $R$-mods.