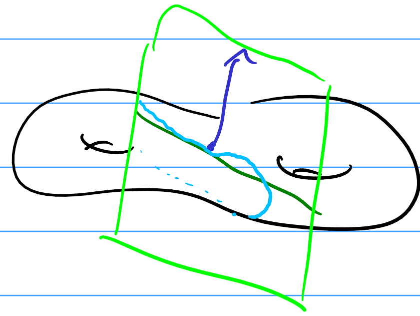
Graded on attendance...
If you miss a day, watch the video,
look at the worksheet,
email me a question.

Goal: Chern - Gauss - Bonnet

Classical Gauss-Bonnet...

$X \subset \mathbb{R}^3$ smooth oriented surface



at $x \in X$, take the unit normal vector
that gives the orientation

for each tangent direction, get a plane
which cuts the surface in a curve

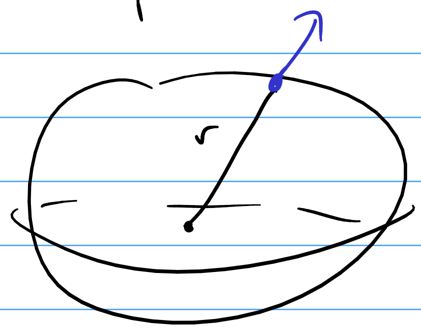
curvature of that curve := $\frac{1}{\text{radius of osculating circle}}$

Signed!

as the tangent direction spins,
get a min and max curvature. "principal curvatures"
(the directions where they occur
turn out to be perpendicular!)

Gauss curvature := min · max.

Examples: round sphere of radius r

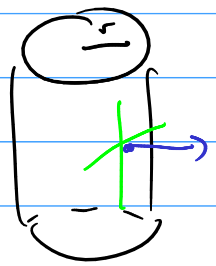


curvature is $\frac{1}{r}$ in every direction.
Gauss curvature is $\frac{1}{r^2}$

$\chi(S^2)$

$$\int_{\text{sphere}} \frac{1}{r^2}\, dA = \frac{1}{r^2} \cdot 4\pi r^2 = 4\pi = 2\pi \cdot 2$$

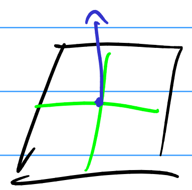
Cylinder



principal curvatures are 0 and $\frac{1}{r}$

Gauss curvature $= 0$

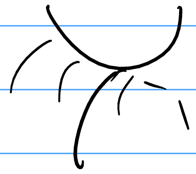
cyl. is (locally) isometric to the plane



here principal curvatures are both 0
Gauss curv. $= 0$

Gauss curvature turns out to be intrinsic.

Hyperboloid $z = x^2 - y^2$



my two cross sections are

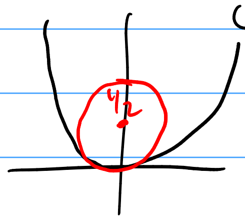
∪ and ∩ $x=0,\ z=-y^2$

$y=0,\ z=x^2$ curvature $= -2$

osculating circle

is $z - z^2 = x^2$

curvature $= 2$

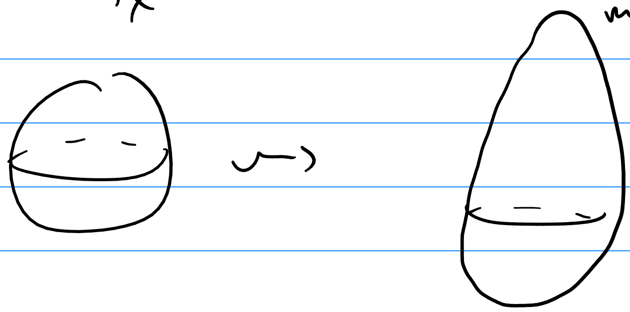


Gauss curvature is -4

Thm: if X is closed

then $\int_X$ curvature $dA = 2\pi \cdot$ Euler char.

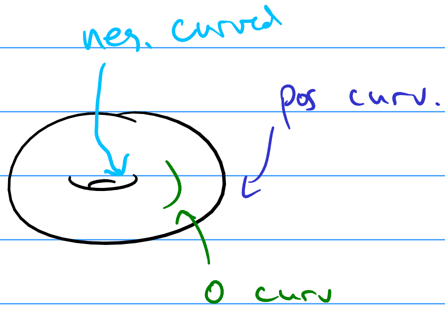
more curvature here



less curvature here

but $\int$ still $= 4\pi$

torus



but $\int$ curvature $= 0$.

How we'll generalize this:

let X = $2n$-dim'l Riemannian manifold

let Ω = Riemann curvature tensor

Ω is a 2-form with values in $End(TX)$ (skew-symmetric)

Pfaffian of Ω is a (real valued) $2n$-form.

$$\int_X \text{Pfaffian of Riem. curvature} = \chi(X)\cdot(2\pi)^n$$

further generalization:

let X be a smooth manifold,

$E \to X$ a vector bundle of rank $2k$ oriented!

choose a Riem. metric on E and a compat. connection

curvature is a 2-form with vals in $End(E)$ again skew-symmetric.

Pfaffian $\rightarrow$ a real valued $2k$-form $\rightsquigarrow$ class in $H^{2k}(X, \mathbb{R})$ indep of choices.

OTOH, a generic section of E vanishes along a submanifold $Z \subset X$ of codim $2k$, with oriented normal bundle

Poincaré dual to the class in $H^{2k}(X, \mathbb{R})$ above

$\forall\, \eta \in H^{\dim X - 2k}(X, \mathbb{R})$,

$$\int_X \eta \wedge \mathrm{Pf}(\Omega) = \int_Y \eta$$

(link: Hopf index thm.)