Graded on attendance...
If you miss a day, watch the video, look at the worksheet, email me a question.

Goal: Chern - Gauss - Bonnet

Classical Gauss - Bonnet...

\( X \subset \mathbb{R}^3 \) smooth oriented surface

at \( \mathbf{x} \in X \), take the unit normal vector that gives the orientation

for each tangent direction, get a plane which cuts the surface in a curve

curvature of that curve: \( \frac{1}{\text{radius of osculating circle}} \)

Signed:

as the tangent direction spins, get a min and max curvature (the directions where they occur turn out to be perpendicular or!)

Gauss curvature := min - max.
Examples: round sphere of radius $r$

Curvature is $\frac{1}{r}$ in every direction.

Gaussian curvature is $\frac{s}{r^3}$

$\int \frac{1}{r} \, dA = \int \frac{1}{r} \cdot 4\pi r^2 = 4\pi = 2\pi \cdot 2$

Cylinder

Principal curvatures are $0$ and $\frac{1}{r}$

Gaussian curvature = 0

Cyl. is (locally) isometric to the plane

Here principal curvatures are both 0

Gaussian curv. = 0

Gaussian curvature turns out to be intrinsic.
Hyperboloid $z = x^2 - y^2$

my two cross sections are

\[ z = 0, \ y = x^2 \]

and

\[ x = 0, \ z = -y^2 \]

curvature = $-2$

osculating circle is $z = -\frac{1}{2}x^2$

curvature = $2$

Gauss curvature is $-4$

Thus: if $X$ is closed

then $\int_X \text{curvature} \ dA = 2\pi \cdot \text{Euler char.}$

more curvature here

less curvature here

but $\int \text{still} = c/\pi$


nels, curved

POS curv.

0 curv.

but $\int \text{curvature} = 0.$
How we'll generalize this:

Let $X = 2n$-dim. Riemannian manifold

Let $R = \text{Riemann curvature tensor}$

$R$ is a 2-form with values in $\text{End}(TX)$ (skew-symmetric)

Pfaffian of $R$ is a

(real valued) 2n-form.

\[
\text{Pfaffian of Riem. curvature } = X(K) \cdot (2\pi)^n
\]

Further generalization:

Let $X$ be a smooth manifold

$E \to X$ a vector bundle of rank $2k$

oriented!

Choose a Riem. metric on $E$

and a conn. connection

curvature is a 2-form with vals in $\text{End}(E)$

again skew-symmetric.

Pfaffian $\to$ a real valued 2k-form

$\to$ class in $H^{2k}(X, \mathbb{Z})$ indep. of choices.
O TOH, a generic section of $E$ vanishes along a submanifold $Z = \mathcal{X}$ of codim $2k$, with oriented normal bundle.

Poincaré dual to the class in $H^{2k}(\mathcal{X},\mathbb{R})$ above $\forall \eta \in H^{d-2k}(\mathcal{X},\mathbb{R})$,

$$\int_{\mathcal{X}} \eta \wedge \text{PP}(\mathcal{X}) = \int_Y \eta$$

(link: Hopf index theorem)