worksheet last time:

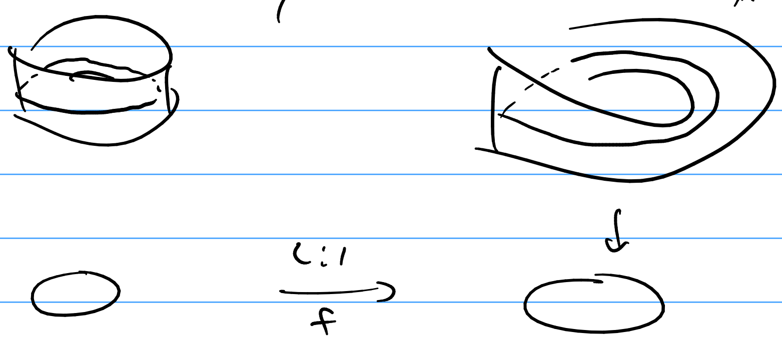
Some authors define a map of v.b's:

$$\begin{tikzcd} E \arrow[r, "g"] \arrow[d, "p"'] & F \arrow[d, "q"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

I would rather call that a map

$g: E \to f^*F$ on one space (X)

pull-back bundle $f^*F = F \times_Y X$. fiber $(f^*F)_x = F_{f(x)}$



$f^*(\text{möbius})$ = trivial bundle.
(two half-twists = no twist)

if $f: X \to Y$
then total derivative is
$Df: TX \to f^*TY$ on X.

A map of vector bundles

$f: E \to F$ (all on X now)

determines linear maps $E_x \to F_x$ $\forall x \in X$

rank of that linear map is upper? semi-cont.

$$\{x \in X \mid \text{rank of } f : E_x \to F_x \leq r\}$$

is closed.

if rank is const. then

$\ker(f)$ $\quad \operatorname{im}(f)$ $\quad \operatorname{coker}(f)$

are vector bundles.

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problem about $C^\infty(X)$-linear maps $\Gamma(E) \to \Gamma(F)$ coming from v.b. maps.

$\exists$ interesting maps $\Gamma(E) \to \Gamma(F)$ that are not $C^\infty(X)$-linear

simplest: let $V \in \Gamma(TX)$ a vector field

determines

$$\Gamma(\mathcal{O}_X) \longrightarrow \Gamma(\mathcal{O}_X)$$
$$g \longmapsto V(g)$$

not $C^\infty(X)$-linear, but rather

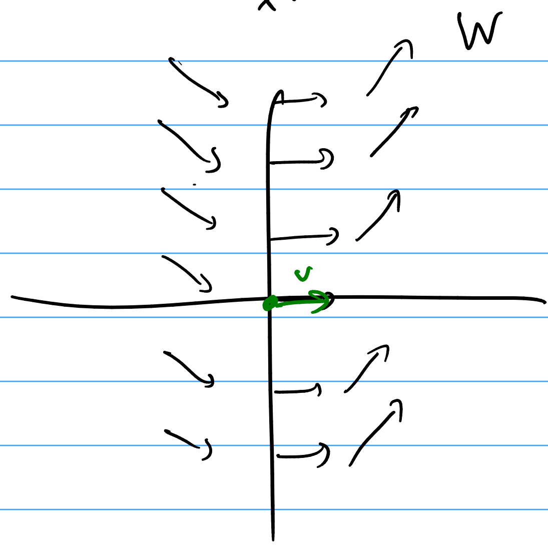
$$V(fg) = V(f)g + fV(g)$$

Connections will be like this.

Connections

Say we're on a mfld X
have a vf W
want the deriv. of W at a point $x \in X$
in a direction $v \in T_x X$

Could try to do it in coords:



e.g. $X = \mathbb{R}^2$

$W = \underset{\sim}{\frac{\partial}{\partial x}} + \underset{\sim}{2x\frac{\partial}{\partial y}}$ (integral curves are parabolas?)

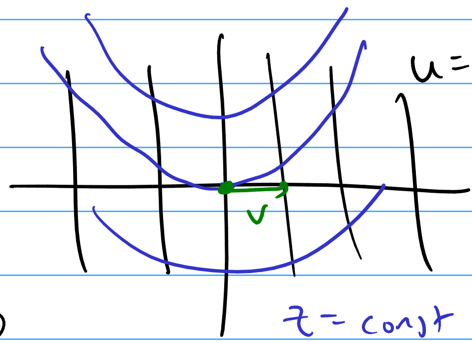
pt $= (0,0)$ $\quad v = \frac{\partial}{\partial x}$

looks like deriv. of W should be $0\frac{\partial}{\partial x} + \boxed{2\frac{\partial}{\partial y}}$?

but if we change vars

$u = x$

$z = y - x^2$



$$\frac{\partial}{\partial x} = \frac{\partial u}{\partial x}\frac{\partial}{\partial u} + \frac{\partial z}{\partial x}\frac{\partial}{\partial z} = \frac{\partial}{\partial u} - 2x\frac{\partial}{\partial z}$$

$$= \frac{\partial}{\partial u} - 2u\frac{\partial}{\partial z}$$

$$v = \left.\frac{\partial}{\partial x}\right|_{(0,0)} = \frac{\partial}{\partial u} - 2\cdot 0\frac{\partial}{\partial z}$$

$$= \left.\frac{\partial}{\partial u}\right|_{(0,0)}$$

$$\frac{\partial}{\partial y} = \frac{\partial u}{\partial y}\frac{\partial}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial}{\partial z} = \frac{\partial}{\partial z}$$

$$W = \underset{\sim}{\frac{\partial}{\partial u} - 2u\frac{\partial}{\partial z}} + \underset{\sim}{2u\frac{\partial}{\partial z}} = \frac{\partial}{\partial u}$$

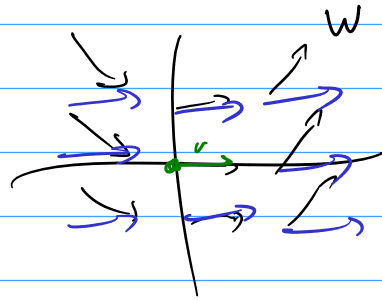
$\longrightarrow$ deriv. of W looks like $\boxed{0}$.

Next try: still have v.f W
and $v \in T_x X$

extend v to a vector field $V \in \Gamma(TX)$

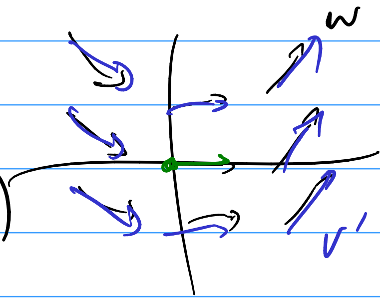
take the flow along V

$$\underline{\Phi}: X \times \mathbb{R}_t \longrightarrow X$$



let W move along this flow
take $\frac{\partial}{\partial t}$, set $t=0$

Lie derivative of W along the v.f V.



fact: get $[V, W]$ (Lie bracket)

issue: different extensions V give different answers.

$\underline{W = \frac{\partial}{\partial x} + 2x \frac{\partial}{\partial y}}$ if $V = \frac{\partial}{\partial x}$ then

$$VW - WV = 2\frac{\partial}{\partial y} - 0 = 2\frac{\partial}{\partial y}$$

but if $V' = \frac{\partial}{\partial x} + 2x\frac{\partial}{\partial y}$ then $V|_{(0,0)} = V'|_{(0,0)}$

$$V'W - WV' = 0.$$

(Lee's smooth manifolds is in Canvas Files.
also Riemannian)

need something new.

A connection on a v.b. E (take $E = TX$ if you want)

is a map

$$\nabla : \Gamma(TX) \otimes_{\mathbb{R}} \Gamma(E) \longrightarrow \Gamma(E)$$

$$V , \quad W \longmapsto \nabla_V W$$

such that ① tensorial in V.

$$\nabla_{fV} W = f \nabla_V W$$

so $\nabla_V W \big|_{\text{point}}$ only depends on $V \big|_{\text{point}}$

② Leibniz rule in W

$$\nabla_V (fW) = V(f) \cdot W + f \cdot \nabla_V W$$

Could repackage as a map $\left(\Omega^1_X = T^*X \right)$

$$\nabla : \Gamma(E) \longrightarrow \Gamma(\Omega^1_X \otimes E)$$

satisfying $\nabla(fW) = df \otimes W + f \cdot \nabla W$

∇W is a 1-form with values in E.

Next time: every vector bundle admits connections.