Last time: X a manifold

$p: E \to X$ a vector bundle.

a connection is $\nabla : \Gamma(TX) \otimes_{\mathbb{R}} \Gamma(E) \to \Gamma(E)$

such that for $v \in \Gamma(TX)$, $s \in \Gamma(E)$, $f \in C^\infty(X)$ we have

$$\nabla_{fv}(s) = f \nabla_v(s)$$

$$\nabla_v(fs) = v(f) \cdot s + f \cdot \nabla_v(s)$$

value of $\nabla_v(s)$ at a point $x \in X$ depends only on $v_x \in T_x X$

and on s in a nbd of x

in fact it only depends on s to first order in the direction v.

Prop: every vector bundle admits a connection.

Pf choose an open cover $\{U_i\}$ of X

and local trivialization $E|_{U_i} \xrightarrow[\varphi_i]{\cong} \mathcal{O}^r_{U_i}$ $(U_i \times \mathbb{R}^r)$

on each U_i, get a connection ∇_i by differentiating components.

given $s: U_i \to U|_{U_i}$ use φ_i to turn it into r functions $(g_1, \dots, g_r): U_i \to \mathbb{R}^r$

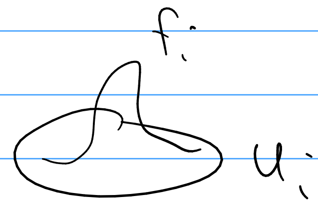
differentiate them: $\left(v(g_1), \dots, v(g_r) \right) : U_i \to \mathbb{R}^r$

use φ_i to turn that back into a section
$\nabla_{i,v}(s) : U_i \to E|_{U_i}$

$\nabla_i|_{U_i \cap U_j}$ does not agree with $\nabla_j|_{U_i \cap U_j}$

let f_i be a partition of 1 subordinate to the cover U_i

(so $f_i : U_i \to [0,1]$ compactly supported,
loc. only fin. many f_i are non-zero
$\sum f_i = 1$)



globally, let $\nabla = \sum f_i \nabla_i$

check: if ∇_1 and ∇_2 are connections
then $t\nabla_1 + (1-t)\nabla_2$ is a connection...

E

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a connection is not a tensor field
but the difference between 2 connections is:
$\nabla_1 - \nabla_2$ is a 1-form with vals in $\mathrm{End}(E)$

$$\nabla_1 - \nabla_2 : \underset{v}{\Gamma(TX)} \otimes_{\mathbb{R}} \underset{s}{\Gamma(E)} \longrightarrow \Gamma(E)$$

still $C^\infty(X)$-linear in the first argument
now also in the second argument:

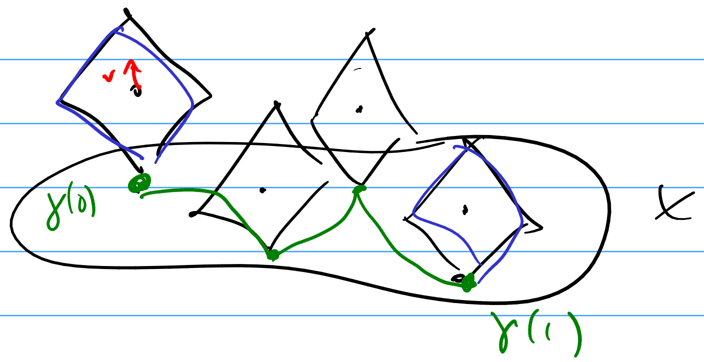
$$\nabla_{1,v}(fs) - \nabla_{2,v}(fs) = \cancel{v(f)\cdot s} + f\nabla_{1,v}(s) - \cancel{v(f)\cdot s} - f\nabla_{2,v}(s)$$
$$= f \cdot (\nabla_{1,v} - \nabla_{2,v})(s)$$

so $\nabla_1 - \nabla_2$ comes from a section of $T^*X \otimes E^* \otimes E$

if ∇ is a connection and $\omega \in \Gamma(\Omega^1_X \otimes \text{End}(E))$
then $\nabla + \omega$ is another connection.

so $\{\text{connections on } E\}$ is an affine space
modeled on the v.s. $\Gamma(\Omega^1_X \otimes \text{End}\, E)$

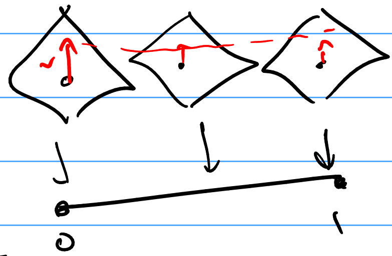
Parallel Transport



given a v.b. $p: E \to X$
and a connection ∇ on E

and piecewise smooth path $\gamma: [0,1] \longrightarrow X$
get a linear map $E_{\gamma(0)} \longrightarrow E_{\gamma(1)}$

how? if γ is smooth,



choose $v \in E_{\gamma(0)}$. take $E|_I$

want a section $s: I \to E|_I$
such that $s(0) = v$
and $\nabla_{\partial/\partial t} s = 0$

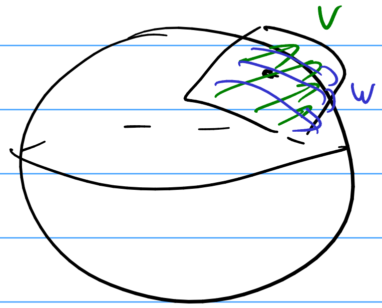
can do it by existence + uniqueness of sols to ODEs.

check: linear in v. uses uniqueness of sols to ODEs.

if γ is only piecewise smooth, do it several times.

BTW this gives a trivialization of $E|_I$.

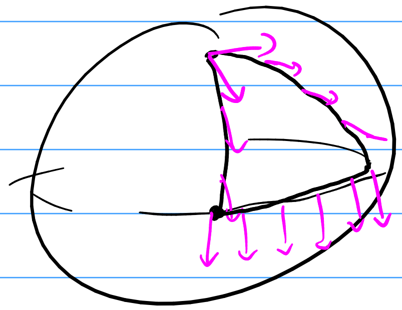
standard example: $S^2 \subset \mathbb{R}^3$



connection on $TS^2 \subset T\mathbb{R}^3|_{S^2} = \mathcal{O}^3_{S^2}$

$\nabla_v(w)$ = differentiate using std connection on $\mathcal{O}^3_{S^2}$,

then orth. project back into TS^2



given a loop based at $x \in X$, get a linear map $E_x \circlearrowleft$

different loops ~> gen. a subgroup of $GL(E) = GL_r(\mathbb{R})$

"holonomy group of the connection"

Conversely, if you've got an assignment

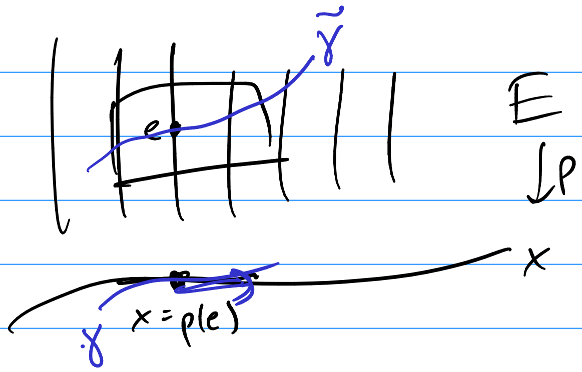
piece smooth paths $\gamma \longmapsto$ linear maps $E_{\gamma(0)} \to E_{\gamma(1)}$

that respects composition of paths and "varies smoothly with γ"

then it comes from a connection.

Worksheet last time:

$$\begin{array}{c} E \\ p \downarrow \\ X \end{array}$$



$$0 \to p^* E \to TE \xrightarrow{Dp} p^* TX \to 0$$

$$p^* TX\big|_e = TX\big|_{p(e)}$$

at a point $e \in E$ with $x = p(e) \in X$,
this is

$$0 \to E_x \to T_e E \to T_x X \to 0$$

$$\tilde{\gamma}'(0) \longleftarrow v$$

a connection ∇ gives a splitting of this seq.

given a tangent vector $v \in T_x X$
and a point $e \in E_x$

choose a path $\gamma : (-\varepsilon, \varepsilon) \to X$ with $\gamma'(0) = v$
use ∇ to parallel transport e along γ
get $\tilde{\gamma} : (-\varepsilon, \varepsilon) \to E$ with $p \circ \tilde{\gamma} = \gamma$
take $\tilde{\gamma}'(0) \in T_e E$

Conversely, given a splitting,
get a connection: next time (or later).