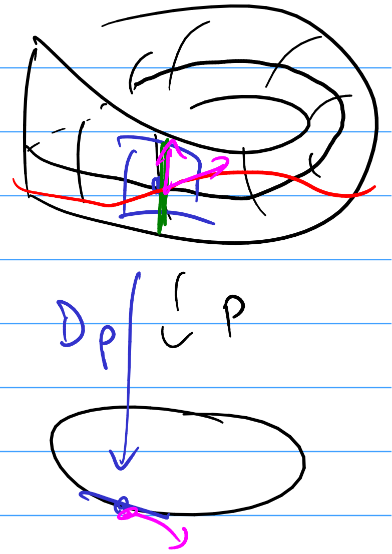
Left over from last time:

$p: E \to X$ a vector bundle



$$0 \to p^*E \to TE \xrightarrow{Dp} p^*TX \to 0$$

Given a splitting of this s.e.s coming from a connection, how to recover the connection?

$$\sigma \in \Gamma(E)$$

$\sigma : X \to E$ $\qquad D\sigma : TX \to \sigma^* TE$

apply σ^* to the exact seq above, get

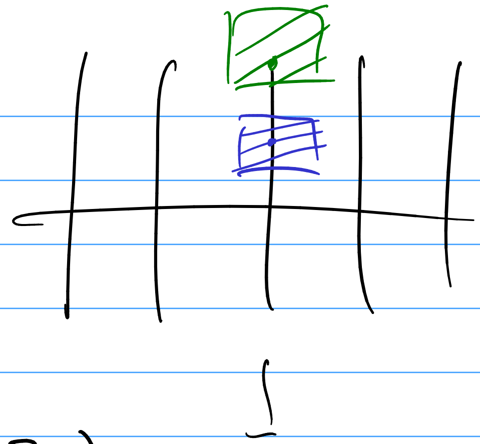
$$0 \to E \to \sigma^* TE \to TX \to 0$$

use splitting $\sigma^* TE \to E$ to turn $D\sigma$ into a map $TX \to E$.

thus $\Gamma(E) \longrightarrow \Gamma(T^*X \otimes E)$

Smoothly, a short exact seq. of vector bundles
$$0 \to F' \to F \to F'' \to 0$$
has lots of splittings. (get 'em locally, use partition of 1)

Torsor over $\mathrm{Hom}(F'', F')$



not every splitting of

$$0 \to p^*E \to TE \to p^*TX \to 0$$

comes from a connection

enough that's equivariant (invariant?) for the action of $\mathbb{R}$ or $\mathbb{C}^*$ that rescales the fibers of E.

(fun if you're interested).

connections are a torsor over $\Gamma(T^*X \otimes \mathrm{End}(E))$

splittings — — over $\Gamma(p^*T^*X \otimes p^*E)$

examples of v.b.'s and sections so far:

tangent bundle TX (vector fields)

cotangent bundle $\Omega_X = \Omega^1_X = T^*X$ (1-forms)

$\Omega^k_X = \Lambda^k \Omega_X$ (k-forms)

Möbius bundle

let's get more examples.

Grassmannians

$V = \mathbb{R}^n$ or $\mathbb{C}^n$

__Gr(k,n)__ $= Gr(k,V) = \{$ k-dim'l subspaces of $V\}$
$= \{$ injections $\mathbb{R}^k$ or $\mathbb{C}^k \hookrightarrow V$ up to change of basis on left. $\}$

$Gr(n,k) = Gr(V,k) = \{$ k-dim'l quotients of $V\}$
$= \{$ surjections $V \twoheadrightarrow \mathbb{R}^k$ or $\mathbb{C}^k$ up to change of basis on right $\}$

$$\underset{\text{quot}}{Gr(n,k)} \underset{\longleftrightarrow}{=} \underset{\text{kernel}}{Gr(n-k,n)}$$

So $Gr(2,5) = Gr(5,3)$

also $Gr(k,V) = Gr(V^*,k)$
by transposing $\mathbb{R}^k \hookrightarrow V$ vs. $V^* \twoheadrightarrow \mathbb{R}^k$

$Gr(5,3) = Gr(3,5)$

gets a topology from the map

$$GL_n \twoheadrightarrow Gr(k,n)$$
$$A \longmapsto A(\text{subspace spanned by } e_1 \ldots e_k)$$
$= $ subspace spanned by first k columns of A

take the quotient top.

restrict to $SO(n) \subset Gl_n \mathbb{R}$
or $U(n) \subset Gl_n \mathbb{C}$

w) still surjective
so $Gr(k,n)$ is compact.

// tautological bundles:

$$S = \{ W \in Gr(k,n) , v \in V \mid v \in W \}$$
$$\downarrow$$
$$Gr(k,n)$$

vector bundle. fiber over $W \in Gr$ is $W \subset V$

$\mathrm{rank}(S) = k$.

also Q = tautological quot. bundle.
fiber over $W \in Gr$ is V/W

rank is $n-k$

exact seq. $0 \to S \to \mathcal{O}_{Gr}^n \to Q \to 0$
or $0 \to S \to \mathcal{O}_{Gr} \otimes V \to Q \to 0$

iso. $Gr(k,V) \cong Gr(V^*, k)$
switches Q with S^*
S with Q^*

special case: $\mathbb{P}^{n-1} = \mathbb{P}V = Gr(1, n)$

then S is called $\mathcal{O}(\underline{-1})$
tautological line bundle.

fun: $S^1 = \mathbb{RP}^1$ Möbius bundle $= \mathcal{O}(-1)$
or $\mathcal{O}(1)$

$S^2 = \mathbb{CP}^1$ Hopf fiber. $S^3 \to S^2$
fiber is S^1. unit circle bundle of
$\mathcal{O}(-1)$

sections of these bundles?

a linear form $\ell \in V^*$
gives a section of S^*

$$\begin{aligned} Gr &\longrightarrow S^* \\ W &\longmapsto \ell|_W \in W^* \end{aligned}$$

when does it vanish?
ℓ is determined (up to scale)
by its kernel $K \cong \mathbb{R}^{n-1} \subset \mathbb{R}^n$

section of S^* vanishes on
$Gr(k, K) \subset Gr(k, V)$
$Gr(k, n-1) \subset Gr(k, n)$

more generally a homogeneous polynomial $f \in \mathrm{Sym}^d V^*$

gives a section of $\mathrm{Sym}^d V^*$ on $Gr(k,n)$

vanishes on $\{ W \in Gr \mid f|_W = 0 \}$

$\left(\text{if } V = \mathbb{R}^n \text{ or } \mathbb{C}^n \text{ then } \mathrm{Sym}\, V^* = \mathbb{R}[x_1 \dots x_n] \text{ or } \mathbb{C}[x_1 \dots x_n] \right)$

special case $k=1$:

$\mathrm{Sym}^d S^* =: \mathcal{O}(d)$

$\mathcal{O}(1) = \mathcal{O}(-1)^*$

$\mathcal{O}(d) = \mathcal{O}(1) \otimes \dots \otimes \mathcal{O}(1)$ d times

$\mathrm{Sym}^d V^* \to \Gamma(\mathrm{Sym}^d S^*)$ gives all the holo sections.

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a p-form $\omega \in \Lambda^p V^*$ gives a section of $\Lambda^p S^*$, $W \mapsto \omega|_W \in \Lambda^p W^*$

vanishes on $\{ W \in Gr \mid \omega|_W = 0 \}$

W is isotropic for ω.

if $p=2$ and $n=2k$ and $\overbrace{\omega \wedge \dots \wedge \omega}^{k \text{ times}} \neq 0$

call this vanishing set Lagrangian Grassmannian $LGr(k,2k) \subset Gr(k,2k)$

similarly, a vector $v \in V$
gives a section of Q

$$W \longmapsto \bar{v} \in V/W$$

vanishes on $\{ W \in Gr \mid v \in W \}$

a d-dim'l subspace $U \subset V$ ... choose a basis $\underline{u_1, \dots u_d} \in U$
$u_1 \wedge \dots \wedge u_d \in \Lambda^d V$
$\rightsquigarrow$ section of $\Lambda^d Q$

vanishes on
$$\{ W \in Gr \mid W \cap U \neq 0 \}$$

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a map $f: X \longrightarrow Gr(n, k)$

= a vector bundle E on X of rank k
with n sections
that don't vanish simultaneously.

aka a surjection $\mathcal{O}^n \twoheadrightarrow E$

(a map $\mathcal{O} \to E$ is the same as a
section of E)

given $f: X \to Gr$, take $\mathcal{O}^n \twoheadrightarrow Q \to 0$
and apply f^*

by partition of 1 argument, every v.b. on
a compact space can be generated by fin. many
smooth sections, so pulled back from $Gr(n, k)$