Associated + Principal Bundles.

Let $E \to X$ be a vector bundle.

Talked about $\mathrm{Sym}^d E$, $\Lambda^d E$ ...

fix an open cover U_i of X
and trivializations

$$\begin{tikzcd} E|_{U_i} \arrow[rr, "\cong", "\varphi_i"'] \arrow[dr] & & U_i \times \mathbb{R}^n \arrow[dl] \\ & U_i & \end{tikzcd}$$

so for $U_i \cap U_j$, get

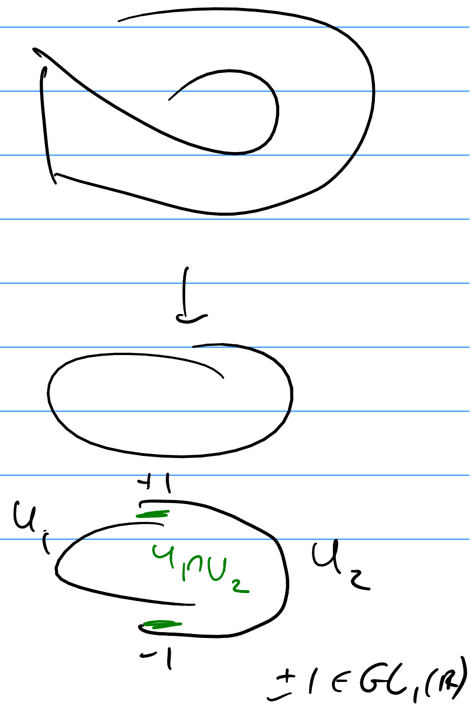
$$(U_i \cap U_j) \times \mathbb{R}^n \xleftarrow[\cong]{\varphi_i} E|_{U_i \cap U_j} \xrightarrow[\cong]{\varphi_j} (U_i \cap U_j) \times \mathbb{R}^n$$

then $\psi_{ij} := \varphi_j \circ \varphi_i^{-1}$ can be repackaged as
a map $U_i \cap U_j \to GL_n(\mathbb{R})$

ψ_{ij}'s tell us how to glue together $\mathbb{R}^n$'s
to get E.

but GL_n also acts on
$\mathrm{Sym}^d(\mathbb{R}^n)$ or $\Lambda^d(\mathbb{R}^n)$

use same ψ_{ij}'s to glue those
together $\rightsquigarrow$ get $\mathrm{Sym}^d E$ or $\Lambda^d E$.



Also E^* is built by taking

$$U_i \cap U_j \xrightarrow{\varphi_{ij}} Gl_n \xrightarrow{T} Gl_n$$

and gluing $\mathbb{R}^n$'s that way.

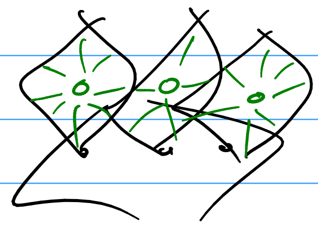
Gl_n also acts on $\mathbb{P}^{n-1}$

use φ_{ij}'s to glue together copies of $(U_i \cap U_j) \times \mathbb{P}^{n-1}$

$\rightsquigarrow$ get $\begin{array}{c} \mathbb{P}E \\ \downarrow \\ X \end{array}$ a $\mathbb{P}^{n-1}$-bundle.

could also describe it as $\left(E \setminus 0 \text{ section}\right) / \mathbb{R}^* \text{ or } \mathbb{C}^*$

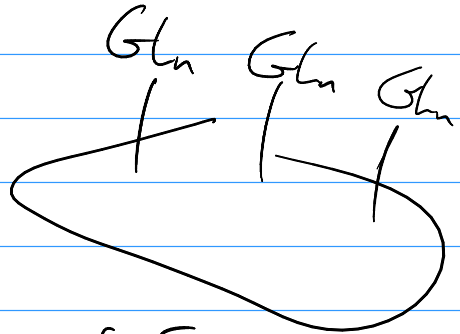
or $\begin{array}{c} Gr(k, E) \\ \downarrow \\ X \end{array}$ a bundle of $Gr(k, n)$



GL_n acts on itself by left multiplication.

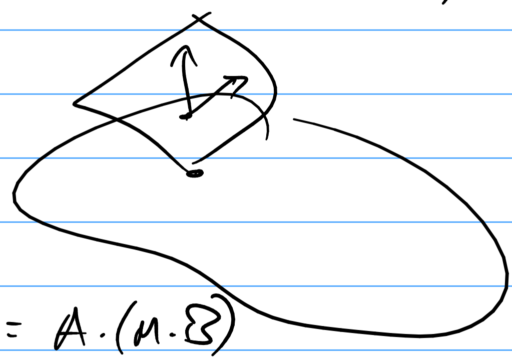
use φ_{ij}'s to make a bundle of GL_n's over X

$$\begin{array}{c} P \\ \downarrow \\ X \end{array}$$



alternate descr: frame bundle of E

$$\left\{ x \in X,\ v_1, \ldots, v_n \in E_x \,\middle|\, v_i\text{'s are lin. indep.} \right\}$$



GL_n also acts on itself on the right

that action commutes with the left action $(A \cdot M) \cdot B = A \cdot (M \cdot B)$

so GL_n still acts on P on the right.

freely, transitively on fibers of $P \to X$.

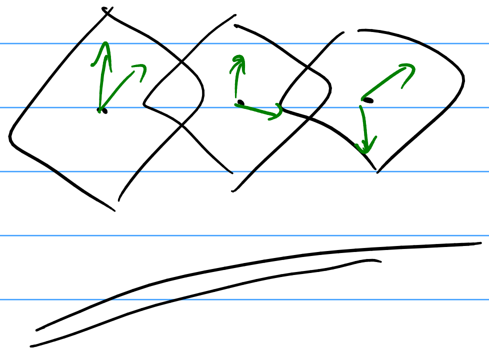
$P/GL_n = X$

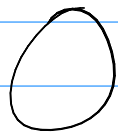
"principal bundle"

Can recover all the other assoc. bundles from P as follows:

if Gl_n acts on a top. space or vector space F on the left, take

$$P \times F \big/ Gl_n$$ where for $A \in Gl_n$, we let $A \cdot (p,f) = (p \cdot A^{-1}, A \cdot f)$?




$Gr(k,n)$

Especially useful when X is a homog. space.

e.g. $X = Gr(k,n)$

$G = Gl_n \circlearrowright X$ transitively, but not freely.

stabilizer of $W_0 = \mathrm{span}(e_1, \dots, e_k) \subset \mathbb{R}$

is
$$H = \begin{array}{c} \begin{array}{cc} k & n-k \end{array} \\ \left(\begin{array}{c|c} * & * \\ \hline 0 & * \end{array}\right) \end{array} \subset G$$

get a map $p: G \twoheadrightarrow Gr$

$$A \longmapsto A \cdot W_0$$

fiber $p^{-1}(W_0) = H$

p is a principal H-bundle.

letting H act on G on left or maybe right, that's free, orbits = stabilizers of different $W \in Gr$

quot G/H is Gr.

So for any rep. of H, get a v.b. on $Gr(k,n)$.

$$\begin{array}{cc} k & n-k \\ \end{array}$$
$$\left(\begin{array}{c|c} * & * \\ \hline 0 & * \end{array}\right)$$

$H \circlearrowright \mathbb{R}^k \rightsquigarrow S$ tautological sub-bundle

$H \circlearrowright \mathbb{R}^{n-k} \rightsquigarrow Q$ taut. quot bundle.

Worksheet: think about $T_{Gr} = S^* \otimes Q$ or $Hom(S, Q)$