Reduction of the Structure Group

Let $p: E \to X$ be a real vector bundle.

A Riemannian metric is a smooth choice of inner product on each fiber E_x, $x \in X$.

$g \in \Gamma(\mathrm{Sym}^2 E^*) \subset \Gamma(E^* \otimes E^*)$

if s, t are two sections then $g(s,t) = g(t,s)$ is a function.

$g(s,s) > 0$ where $s \neq 0$.

Prop: Every bundle admits Riem. metrics.
Pf: do it locally, partition of 1 argument.
weight average of two inner products is an inner product. $\square$

Given an open cover $\{U_i\}$ of X and trivializations $\varphi_i : E|_{U_i} \to \mathcal{O}_{U_i}^n$

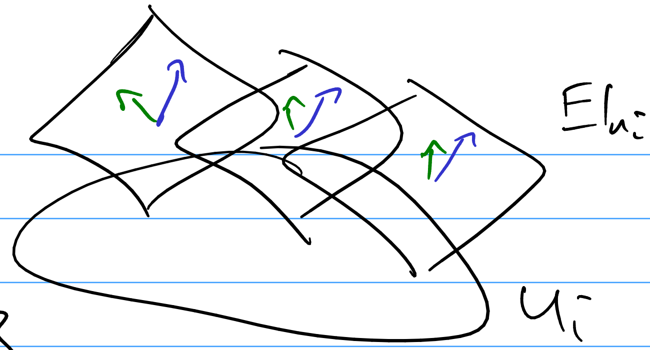
get $\psi_{ij} = \varphi_j^{-1} \circ \varphi_i : \mathcal{O}^n_{U_i \cap U_j} \to \mathcal{O}^n_{U_i \cap U_j}$

aka functions with values in $GL_n(\mathbb{R})$

Given a metric, take the φ_i's and do Gram-Schmidt to get trivializations

$$E|_{U_i} \xrightarrow{\cong} \mathcal{O}^n_{U_i}$$

Such that g agrees with the standard inner product $\langle e_i, e_j \rangle = \delta_{ij}$.



Then the transition maps ψ_{ij} take values in

$$O(n) = \{A \in GL_n(\mathbb{R}) \mid A^T A = 1\}$$

(orthogonal group)

then we get a principal $O(n)$-bundle over X

$$\begin{array}{l} P = \{x \in X, \text{ orthonormal basis for } E_x\} \\ \downarrow \\ X \end{array}$$

Conversely, if E is associated to a principal $O(n)$-bundle over X (std rep. of $O(n)$ on $\mathbb{R}^n$) then E gets a Riem. metric.

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a volume form on E is a non-vanishing section of $\underline{\underline{\Lambda^{top} E^*}}$ (line bundle)

subgroup of GL_n preserving the std. vol. form on $\mathbb{R}^n$ is $SL_n = \{A \in GL_n \mid \det A = 1\}$
special linear group

an orientation of E is a section of the double cover of X assoc. to the map

$$GL_n(\mathbb{R}) \longrightarrow \{\pm 1\}$$
$$A \longmapsto \text{sign of } \det A$$

Same as a reduction of the str. group to
$GL_n^+(\mathbb{R}) := \{ A \in GL_n \mid \det A > 0 \}$
(matrices preserving the std orientation of $\mathbb{R}^n$)

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a symplectic form on E of rank $2n$
is $\omega \in \Gamma(\Lambda^2 E^*)$
such that $\omega \wedge \cdots \wedge \omega \in \Gamma(\Lambda^{2n} E^*)$
never vanishes.

equivalently, the map $E_x \longrightarrow E_x^*$, $v \longmapsto \omega(v,-)$ is an iso $\forall x \in X$

standard symplectic form on $\mathbb{R}^{2n}$ is

$$\Omega \text{ or } J = \left(\begin{array}{c|c} 0 & -1 \\ \hline 1 & 0 \end{array}\right)$$

aka $dx_1 \wedge dx_2 + dx_3 \wedge dx_4 + \cdots + dx_{2n-1} \wedge dx_{2n}$.

$\exists$ a Gram-Schmidt process
that takes any basis for E_x
and turns it into one where ω looks like J.

Symplectic form = same as reduction from GL_{2n} to
$$Sp_{2n}(\mathbb{R}) = \{ A \in GL_n \mid A^T J A = J \}$$
(non-compact) symplectic group.

For $G \subset GL_n$, and a rank-n vector bundle E
a reduction of the structure group to G
is a choice of transition maps
ψ_{ij} with values in $G \subset GL_n$
up to some notion of equivalence...

or a principal G-bundle

$$\begin{array}{c} P \\ \downarrow \\ X \end{array}$$

(G acts on P, freely + transitively on fibers.)

such that the assoc. $\mathbb{R}^n$-bundle is E.

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a complex str. on E

is a map $J \in \Gamma(\mathrm{Hom}(E,E))$

such that $J^2 = -1$

Same as a reduction to

$$GL_n\mathbb{C} = \left\{ A \in GL_{2n}(\mathbb{R}) \;\middle|\; AJ = JA \right\}$$

where $J = \left(\begin{array}{c|c} 0 & -1 \\ \hline 1 & 0 \end{array}\right)$.