Last time: $G < Gl_n(\mathbb{R})$

$p: E \to X$ a rank-n vector bundle

a reduction of the str. group from Gl_n to G is either:

- choice of transition maps
  $\varphi_{ij} : U_i \cap U_j \to G < Gl_n$

  up to equivalence...

- or a principal G-bundle $P \to X$
  and an iso $P \times_G \mathbb{R}^n \cong E$

for familiar groups, correspond to familiar geom. structures on E:

$O(n) \longleftrightarrow$ Riem. metric $g \in \Gamma(\mathrm{Sym}^2 E^*)$

$SL_n(\mathbb{R}) \longleftrightarrow$ volume form in $\Gamma(\Lambda^n E^*)$

$Gl_n^+(\mathbb{R}) \longleftrightarrow$ orientation

$Sp_{2n}(\mathbb{R}) \longleftrightarrow$ symplectic form $\omega \in \Gamma(\Lambda^2 E^*)$
(rank E should be even)

$Gl_n(\mathbb{C}) \subset Gl_{2n}(\mathbb{R}) \longleftrightarrow$ complex structure
$J \in \Gamma(\mathrm{End}(E))$
$J^2 = -1$

$O(n) \cap Gl_n^+ = SO(n) \subset Sl_n$

if $A^T A = 1$ then $\det(A)^2 = 1$

so $\det(A) = \pm 1$

geom: Riem. metric + orientation $\rightsquigarrow$ volume form

write g in coords as

$$g = \sum g_{ij}\, dx^i \cdot dx^j$$

then $d\mathrm{Vol} = \sqrt{\det(g_{ij})} \cdot dx_1 \wedge \cdots \wedge dx_n$

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for G-structures on the tangent bundle TX, we can ask for more.

A G-structure on TX is integrable if

we can take the open sets U_i to be coordinate patches

and trivializations $\varphi_i : TX|_{U_i} \cong \mathcal{O}_{U_i}^n$

to come from coordinates

$$\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n} \in \Gamma(TX|_{U_i})$$

so the transition maps

φ_{ij} are $\left(\frac{\partial y^k}{\partial x^l}\right)$ taking values in $G \subset Gl_n(\mathbb{R})$

Fave example: $G = GL_n(\mathbb{C}) \subset GL_{2n}(\mathbb{R})$

$$= \left\{ A \;\middle|\; AJ = JA \right\} \qquad J = \left(\begin{array}{c|c} 0 & -1 \\ \hline 1 & 0 \end{array}\right)$$

$$= \left\{ \begin{pmatrix} B & -C \\ C & B \end{pmatrix} \right\}$$

$\hookrightarrow$ Corresp to $B + iC$

if our cx str. is not (necessarily) integrable, call it an <u>almost cx str.</u>

integrable $\Leftrightarrow$ (actual) complex str.
X is a complex manifold.

choose coords $x_1 \cdots x_n, y_1 \cdots y_n$ on U_i

and $x_1', \cdots x_n', y_1' \cdots y_n'$ on U_j

transition maps $\left(\begin{array}{c|c} \frac{\partial x_k'}{\partial x_\ell} & \frac{\partial y_k'}{\partial x_\ell} \\ \hline \frac{\partial x_k'}{\partial y_\ell} & \frac{\partial y_k'}{\partial y_\ell} \end{array}\right)$ if this is $\begin{pmatrix} B & -C \\ C & B \end{pmatrix}$

then $\dfrac{\partial x_k'}{\partial x_\ell} = \dfrac{\partial y_k'}{\partial y_\ell}$ and $\dfrac{\partial y_k'}{\partial x_\ell} = -\dfrac{\partial x_k'}{\partial y_\ell}$

Cauchy-Riemann equations! $x_k' + i y_k'$ are holo functions of $x_k + i y_k$.

Thm: J is integrable iff condition with Lie brackets and J. (Nijenhuis)

Symplectic structures:

$Sp_{2n}(\mathbb{R})$ - structure on TX
is integrable iff
the symplectic form $\omega \in \Gamma(\Lambda^2 T^*X)$
sat. $d\omega = 0$.

why? standard 2-form $dx_1 \wedge dx_2 + \cdots + dx_{2n-1} \wedge dx_n$
is closed.
if can choose coords so that $\left(\frac{\partial y_k}{\partial x_\ell}\right)$

preserve the std form, then we get a global form, still with $d\omega = 0$.

Converse: given a non-degen. 2-form with $d\omega = 0$, can get coords in which it looks like the standard one. (Darboux's theorem)

Orientations are always integrable:
take some random coords $x_1, \ldots, x_n$

if $\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right)$ is positively oriented, then happy

else $x_1 \longrightarrow -x_1$ fixes the problem.

Similarly with a volume form:
choose any coords. then
$$dx_1 \wedge \cdots \wedge dx_n = \text{function} \cdot d\mathrm{Vol}$$

rescale x_1 to turn function into 1.

Riemannian metric is integrable iff it's flat.

one direction: the std Riem. metric
$$dx_1^2 + \cdots + dx_n^2$$
is flat.

converse: work.