Remark: to show that $GL_n\mathbb{C} \subset GL_{2n}^+\mathbb{R}$,

$$\begin{tikzcd}
GL_n\mathbb{C} \arrow[r, "\det_{\mathbb{R}}"] \arrow[d, "\det_{\mathbb{C}}"'] & \mathbb{R}^* = \mathbb{R}^+ \cup \mathbb{R}^- \\
\mathbb{C}^* \arrow[ur, dashed, "\exists!"'] &
\end{tikzcd}$$

if we show that $\det_{\mathbb{C}}: GL_n\mathbb{C} \to \mathbb{C}^*$
is the Abelianization,

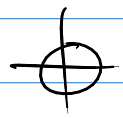
aka $SL_n\mathbb{C}$ is the commutator subgroup of $GL_n\mathbb{C}$
clearly $SL_n\mathbb{C} \subset$ comm. subgroup.
$\det(ABA^{-1}B^{-1}) = 1$

Alternatively, could prove that $GL_n\mathbb{C}$ is connected thu.

① $U(n) = \{A \in GL_n\mathbb{C} \mid A^\dagger A = 1\}$ $\quad A^\dagger = \bar{A}^T$ dagger
unitary group preserves $h(v,w) = \bar{v}^T w$
std. Hermitian metric.

$GL_n\mathbb{C}$ deformation retracts onto $U(n)$
by Gram-Schmidt algorithm.

$U(n)$ is connected by induction:
$U(1) =$ circle



$U(n) \circlearrowright S^{2n-1} \subset \mathbb{C}^n$
stabilizer of $\begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ is $\left\{ \begin{array}{c|c} 1 & 0 \\ \hline 0 & A \end{array} \right\}$

where $A \in U(n-1)$

So we get a fibration

$$\begin{array}{ccc} U(n-1) \hookrightarrow & U(n) & \qquad A \\ & \downarrow & \qquad \downarrow \\ & S^{2n-1} & \qquad A\cdot\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix} \end{array}$$

with connected base + connected fiber
$\rightsquigarrow$ $U(n)$ is connected.

long exact seq. in homotopy groups:

$$\pi_{2n-2}(U(n-1)) \to \pi_{2n-2}(U(n)) \to \cancel{\pi_{2n-2}(S^{2n-1})}\ \pi_{2n-1}(S^{2n-1})$$
$$\to \cdots$$
$$\to \pi_1(U(n-1)) \to \pi_1(U(n)) \to \cancel{\pi_1(S^{2n-1})}$$
$$\to \pi_0(U(n-1)) \to \pi_0(U(n)) \to \cancel{\pi_0(S^{2n-1}) \to 0}$$

find that inclusion $U(n-1) \hookrightarrow U(n)$

$$\begin{pmatrix} 1 & \\ & * \end{pmatrix}$$

induces an iso on $\pi_{\leq 2n-3}$

and a surjection on π_{2n-2}

[... Bott periodicity is further down this road.]

$\pi_1(U(n)) = \mathbb{Z}$, $\pi_1(O(n)) = \mathbb{Z}/2$ $\forall n \geq 2$

For a Lie group $G \subset Gl_n \mathbb{R}$,

the Lie algebra $\mathfrak{g}$ is $T_1 G$

$\mathfrak{gl}_n \mathbb{R} = T_1 Gl_n \mathbb{R}$ = all $n \times n$ matrices

so $\mathfrak{g} \subset \mathfrak{gl}_n \mathbb{R}$

for $SO(n) = \left\{ A \in Gl_n \mathbb{R} \;\middle|\; \begin{array}{l} A^T A = 1 \\ \det A = 1 \end{array} \right\}$

let $X \in so(n)$

choose a path $A(t)$ in $SO(n)$
with $A(0) = 1$ $A'(0) = X$

then $A^T(t) \cdot A(t) = 1$

$$A'^T(t) A(t) + A^T(t) A'(t) = 0$$

$t=0$

$$X^T \cdot 1 + 1 \cdot X = 0$$
$$X^T = -X$$

other way: given X with $X^T = -X$, take
$A(t) = \exp(tX)$ stays in $SO(n)$

---

mod t^2, just write $A(t) = 1 + tX$

$A^T A = 1 \rightsquigarrow (1 + tX)^T (1 + tX) = 1$

$$\cancel{1} + tX^T + tX + \cancel{t^2 X^T X} = \cancel{1}$$

$X^T X = 0$

One more reduction of str. group.

$$P = \begin{array}{c|c} & \begin{array}{cc} m & n \end{array} \\ \begin{array}{c} m \\ n \end{array} & \left(\begin{array}{c|c} A & B \\ \hline 0 & C \end{array}\right) \end{array} \subset GL_{m+n}(\mathbb{R})$$

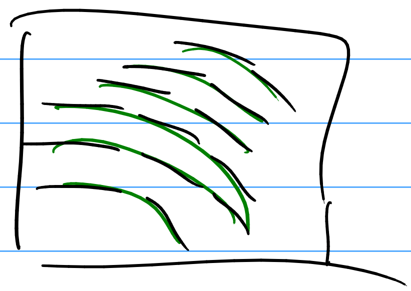
subgroup preserving the subspace spanned by $e_1, \ldots, e_m$

reduction of str. group to P

$\longleftrightarrow$ sub-bundle $F^m \subset \underline{E}^{m+n}$

if $\underline{E} = TX$, this is called a "distribution"

integrable in Wednesday's sense iff it comes from a foliation



Frobenius thm: integrable iff
$\forall\, v, w \in \Gamma(\text{sub-bundle of } TX)$, $[v, w]$ stays in $\Gamma(\text{sub-bundle}) \subset \Gamma(TX)$

applications to PDE
$\hookrightarrow$ Ch. 19 of Lee's Smooth Manifolds.

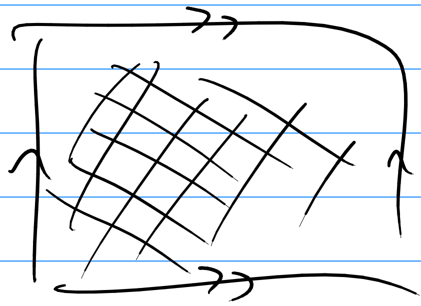
reduction to $GL_n \times GL_m \subset GL_{m+n}$

$$\left(\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right)$$

$\rightsquigarrow$ splitting $E^{m+n} \cong F^m \oplus G^n$

for $E = TX$, does integrable mean $X = Y \times Z$?

not in general: take $X =$ torus



BUT it's true if $\pi_1(X) = 0$. (I think?)

Recall connections:

$$\nabla : \underline{\Gamma(TX)} \otimes \Gamma(E) \longrightarrow \Gamma(E)$$

or $\nabla : \Gamma(E) \longrightarrow \Gamma(T^*X \otimes E)$

$$\nabla(fs) = df \otimes s + f \cdot \nabla s$$

Suppose that E carries a Riem. metric g

then ∇ is compatible with g if

for $s, t \in \Gamma(E)$ we have

$$d \underbrace{g(s,t)}_{\in C^\infty(X)} = g(\nabla s, t) + g(s, \nabla t)$$

Alternatively, ∇ on E induces a connection on $\mathrm{Sym}^2 E$

via $(\nabla g)(s,t) = g(\nabla s, t) + g(s, \nabla t) - d g(s,t)$

and we want $\nabla g = 0$.

or maybe minus this?

Goal: then the connection form takes values in $\mathfrak{so}(n) \subset \mathfrak{gl}_n \mathbb{R}$

later, so does curvature F_∇.

What is the connection form?

locally on X

choose coordinates $x_1, \dots, x_n$

and sections $e_1, \dots, e_r$ of E

that give an orthonormal trivialization

$$\nabla_{\partial/\partial x^i}(e_j) = \sum_{k=1}^{r} \Gamma_{ij}^k e_k \qquad \Gamma_{ij}^k \text{ are functions. Christoffel symbols.}$$

or $$\nabla(e_j) = \underbrace{\sum_{i=1}^{n} \sum_{k=1}^{r} \Gamma_{ij}^k \, dx^i \otimes e_k}$$

the connection form.

$$\omega_j^k = \sum_i \Gamma_{ij}^k dx^i$$

$\omega_j^k = \sum_i \Gamma_{ij}^k$ is a matrix-valued 1-form.

if ∇ is compat with g then
it takes values in $\mathfrak{so}(r) \subset \mathfrak{gl}_r(\mathbb{R})$

See it:

$$g(e_i, e_j) = \delta_{ij}$$

$$d\, g(e_i, e_j) = 0 = g(\nabla e_i, e_j) + g(e_i, \nabla e_j)$$

$$= g\Big(\sum_k \omega_i^k e_k, e_j\Big) + g\Big(e_i, \sum_\ell \omega_j^\ell e_\ell\Big)$$

$$= \sum_k \omega_i^k \delta_{kj} + \sum_\ell \omega_j^\ell \delta_{i\ell}$$

$$= \omega_i^j + \omega_j^i$$

so $\omega_j^i = -\omega_i^j$.

Warning: if $E = TX$,
my orthonormal frame e_i
is probably not $\partial/\partial x^i$
for any choice of coordinates.

Next time: connections on principal G-bundles...
Soon: curvature.