Plan: Mon: Levi-Civita connection
Wed: Chern connection
Fri: connections on principal bundles
next week: curvature.

Levi-Civita connection

not Cività
C ⇝ ch, not s

last time: if a bundle E carries a Riem. metric g, can ask for your connection ∇ on E to be compat. with g.

$$d\, g(s,t) = g(\nabla s, t) + g(s, \nabla t)$$

given a metric g, compat. connections exist by partition of 1 argument.

not unique.

But if $E = TX$ then there is a preferred ∇ compatible with g

for connections on TX in general, have the torsion of the connection

for $V, W \in \Gamma(TX)$ vector fields

$$\tau(V,W) = \nabla_V W - \nabla_W V - [V,W]$$

τ is a 2-form with vals in TX

need to check: $\tau(fV, W) \overset{\checkmark}{=} f\,\tau(V, W) \overset{\text{similar.}}{=} \tau(V, fW)$

$$\tau(fV, W) = \nabla_{fV} W - \underline{\nabla_W(fV)} - [fV, W]$$

$$= f\nabla_V W \underline{- \cancel{W(f)\cdot V} - f\cdot\nabla_W V}$$

$$- f[V, W] + \cancel{W(f)V}$$

as desired.

$fV(W(g)) - W(fV(g))$

$= fVWg - W(f)\,V(g) - f\,WV(g)$

$= f[V, W](g) - W(f)\cdot V(g)$

∇ is torsion-free if $\tau = 0$

[Thm: $\exists!$ torsion-free connection compat with g.]

In coordinates: $[\partial/\partial x^i, \partial/\partial x^j] = 0$, so

if $\nabla_{\partial/\partial x^i}(\partial/\partial x^j) = \sum_k \Gamma_{ij}^k\, \partial/\partial x^k$

then $\tau = 0$ means $\Gamma_{ij}^k = \Gamma_{ji}^k$

"symmetric"

Pf of uniqueness: let $U, V, W \in \Gamma(TX)$

compat' with g:

$$U\, g(V,W) = g(\nabla_U V, W) + g(V, \nabla_U W)$$

uses $\tau = 0$

$$= g(\nabla_U V, W) + g(V, \nabla_W U) + g(V, [U,W])$$

V, W are vector fields

$g(V,W)$ is a function

U(function) is another function

cancel

cyclically permuting:

$$+\{\ V\, g(W,U) = g(\nabla_V W, U) + g(W, \nabla_U V) + g(W, [V,U])$$

$$-\{\ W\, g(U,V) = g(\nabla_W U, V) + g(U, \nabla_V W) + g(U, [W,V])$$

cancel

$$U\, g(V,W) + V\, g(W,U) - W\, g(U,V)$$
$$= 2g(\nabla_U V, W) + g(V, [U,W]) + g(W, [V,U]) - g(U, [W,V])$$

$$g(\nabla_U V, W) = \tfrac{1}{2}(\text{6 terms not involving } \nabla)$$

g is non-degenerate, so we can solve for $\nabla_U V$

uniqueness is done.

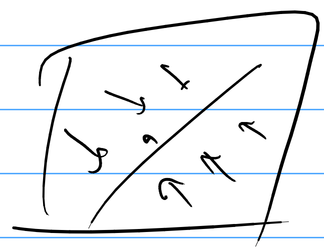
gives a formula for g

in coordinates, $\Gamma_{ij}^k = \frac{1}{2} \sum_\ell g^{k\ell} \left(\frac{\partial g_{j\ell}}{\partial x^i} + \frac{\partial g_{i\ell}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^\ell} \right)$

where $g^{k\ell}$ is the inverse matrix of g_{ij}

check: this gives a torsion-free connection that's compat with g. $\boxed{\checkmark}$

if E is a Riem. vector bundle,



and $F \subset E$ is a sub-bundle,

let $r: E \to F$ be orth. projection

if ∇ is a connection on E compat with g
then $r \circ \nabla$ is a connection on F
compat with $g|_F$

if we had $\varphi: X \hookrightarrow Y$
$E = \varphi^* TY$
$F = TX$

if ∇ on TY was torsion-free
then so is $r \circ \nabla$ on TX.