my earlier claim about
integrable $GL_n \times GL_n$ - structures on TX
may not be right... need compactness (or something)
at least, otherwise
take $X = Y \times Z$ - point

↓

OK when Riemannian metric around
De Rham decomposition thm.

=

Chern Connection

first: (1,0)-forms and (0,1)-forms on cx manifolds.

X = cx manifold
U = open set with
cx coordinates $z_1, \dots, z_n : U \hookrightarrow \mathbb{C}^n$
or real coordinates $z_i = x_i + i y_i$ $\bar{z}_i = x_i - i y_i$

complex-valued
1-forms $\omega = \sum \alpha_i dx_i + \sum \beta_j dy_j$
better basis: $dz_i = dx_i + i dy_i$
$d\bar{z}_i = dx_i - i dy_i$

write $\omega = \underbrace{\sum \omega_i dz_i}_{(1,0) \text{ part of } \omega} + \underbrace{\omega_{\bar{i}} d\bar{z}_i}_{(0,1) \text{ part}}$

exercise: go back and forth

a function $f \in C^\infty(U, \mathbb{C})$ is holomorphic if
df is all type (1,0), no type (0,1)

$$df = \sum \frac{\partial f}{\partial x_i} dx_i + \sum \frac{\partial f}{\partial y_j} dy_j$$

$$= \underbrace{\boxed{\sum \frac{\partial f}{\partial z_i} dz_i}}_{\partial f} + \underbrace{\boxed{\sum \frac{\partial f}{\partial \bar z_i} d\bar z_i}}_{\bar\partial f}$$

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \qquad \frac{\partial}{\partial \bar z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

$$\underline{\frac{\partial f}{\partial x}} dx + \underline{\frac{\partial f}{\partial y}} dy = \frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial \bar z} d\bar z$$

$$= \frac{\partial f}{\partial z}(dx + i\,dy) + \frac{\partial f}{\partial \bar z}(dx - i\,dy)$$

$$= \left(\underline{\frac{\partial f}{\partial z} + \frac{\partial f}{\partial \bar z}}\right) dx + i\,\underline{\left(\frac{\partial f}{\partial z} - \frac{\partial f}{\partial \bar z}\right)} dy$$

f holomorphic $\Longleftrightarrow \bar\partial f = 0 \Longleftrightarrow$ Cauchy-Riemann eqs.

if w_i are some other complex coordinates
w's are holo functions of z's
and vice versa

then decomposition of 1-forms
into $(1,0)$ and $(0,1)$
will be preserved by
change of coords.

$$dw_i = \sum \frac{\partial w_i}{\partial z_j} dz_j + \sum \cancel{\frac{\partial w_i}{\partial \bar z_j}} d\bar z_j$$

$$d\bar w_i = \sum \cancel{\frac{\partial \bar w_i}{\partial z_j}} dz_j + \sum \frac{\partial \bar w_i}{\partial \bar z_j} d\bar z_j$$

So we can talk about $(0,1)$ and $(1,0)$ forms globally

A holomorphic vector bundle

$p: E \to X$ $\quad$ E, X are both cx manifolds

p is holomorphic

+ usual thing about fibers being vector spaces.

$E|_U \cong U \times \mathbb{C}^r$ holo'ly.

equivalently, can get trivializations so the transition maps $\varphi_{ij}: U_i \cap U_j \to GL_r(\mathbb{C})$ are holo functions.

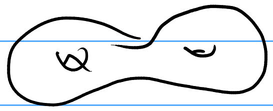
Smooth vector bundles = f.g. proj. modules over $C^\infty(X)$

holo v.b.'s $\neq$ modules over ring of holo functions (if X is compact, this is $\mathbb{C}$)

One C^∞ cx vector bundle may admit many holo. structures.

May vary continuously.

e.g. if $X = \overset{\text{compact}}{\vee}$ Riem. surface of genus g



the trivial l.b. $X \times \mathbb{C}$ admits a $2g$-dim'l torus worth of holo structures. (Jacobian of C)

$$\| $$

$$\mathbb{C}^g / \mathbb{Z}^{2g}$$

if E is a holomorphic v.b.
and $s \in \Gamma(E)$ a smooth section

then ds still doesn't make sense
but $\bar\partial s$ does make sense globally.
it's a $(0,1)$-form with values in E.

So we have half of a connection for free.

details. on an open set $U \subset X$
let $e_1, \dots, e_r \in \Gamma(E|_U)$ and $f_1, \dots, f_r \in \Gamma(E|_U)$
be two holo trivializations of $E|_U$

$f_i = \sum \varphi_i^j e_j$ $\quad$ φ_i^j are holo functions on U.

Section $s = \sum s_i e_i$ $\quad$ s_i smooth functions on U
$= \sum t_i f_i$

try to define $ds = \sum ds_i e_i$ $\quad$ 1-form w/ values in E
$\partial s = \sum \partial s_i e_i$ $\quad$ $(1,0)$-form
$\bar\partial s = \sum \bar\partial s_i e_i$ $\quad$ $(0,1)$-form.

how are s_i and t_i related?

$\sum_i s_i e_i = \sum_i t_i f_i = \sum_{i,j} t_i \varphi_i^j e_j$

$s_i = \sum t_k \varphi_i^k$ $\quad$ d and ∂ have trouble
but $\bar\partial s_i = \sum \bar\partial t_k \varphi_i^k + \cancel{t_k \bar\partial \varphi_i^k}$
because φ_i^k is holo.

So ∂s looks the same in either triv:

$$\sum \partial s_i e_i = \sum \partial t_i f_i$$

Related: on a smooth, real vector bundle
if we can get our transition functions
to be loc. const.
then ds is ok.
get a (flat!) connection for free.

thm: given a Hermitian metric h on E,
$\exists$ unique compat connection
s.t. $(\nabla s)^{0,1} = \bar\partial s$.

for $s \in \Gamma(E)$ smooth

$\nabla s \in \Gamma(T^*X \otimes E)$ 1-form w/ vals in E

if X is complex, get $\nabla^{0,1}$ and $\nabla^{1,0}$

Hermitian metric:
on each fiber,
$h(v,w)$ is $\mathbb{C}$-linear in v, conj.-linear in w
$h(w,v) = \overline{h(v,w)}$
$h(v,v) > 0$ when $v \neq 0$

std. example: on $\mathbb{C}^n$,

$$h(\vec{z}, \vec{w}) = \sum_i \bar{z}_i w_i$$

if ∇ is compat with h, then for two sections s,t, have

$$dh(s,t) = h(\nabla s,t) + h(s,\nabla t)$$

$$\partial h(s,t) = h(\nabla^{1,0}s,t) + h(s,\overset{\bar\partial}{\boxed{\nabla^{0,1}}}t)$$

$$\bar\partial h(s,t) = h(\overset{\bar\partial}{\boxed{\nabla^{0,1}}}s,t) + h(s,\nabla^{1,0}t)$$

h is invertible, so

$$h(\nabla^{1,0}s,t) = \partial h(s,t) - h(s,\bar\partial t)$$

determines $\nabla^{1,0}s$.