Last time: X = cx manifold
E = holo vector bundle
for $s \in C^\infty(E)$ smooth section,
$\bar\partial s$ is well-defined although ∂s is not.

if h is a herm. metric on E
then we get the Chern connection
$\nabla = \nabla^{1,0} + \nabla^{0,1}$
$\nabla^{0,1} \hookrightarrow$ this should be $\bar\partial$

determined by
$$d(h(s,t)) = h(\nabla s, t) + h(s, \nabla t)$$
$$\Rightarrow \partial(h(s,t)) = h(\nabla^{1,0} s, t) + h(s, \bar\partial t)$$

in a holo trivialization, $\bar\partial s = \bar\partial t = 0$ so
$$\partial h(s,t) = h(\nabla s, t) + 0$$

so $\nabla = h^{-1}\partial h$ thinking of h as a matrix-valued function.

also a nice formula in a unitary trivialization

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seen that $GL_n\mathbb{C} \cap O(2n)$ in $GL_{2n}\mathbb{R}$
is $U(n) = \{ A \in GL_n\mathbb{C} \mid \bar{A}^T A = 1 \}$

so a herm. metric on a cx v.b. of rank n
gives a Riem. metric the underlying real
bundle of rank $2n$

if $E = TX$, does the Chern connection of J and h agree with the Levi-Civita connection of g.

Theorem: Yes iff J is parallel for Levi-Civita connection
iff g, J are Kähler

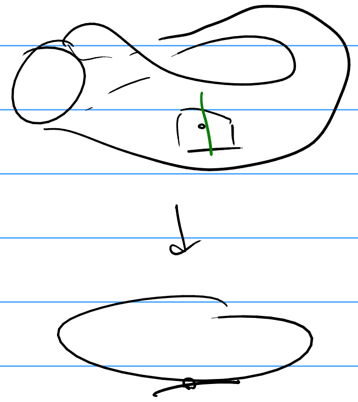
Kähler: the non-degen 2-form
$\omega(-,-) = g(-, J-)$
is closed.

every sm. proj. variety $\subset \mathbb{CP}^n$
is Kähler

Connections on Fiber Bundles

for any fiber bundle

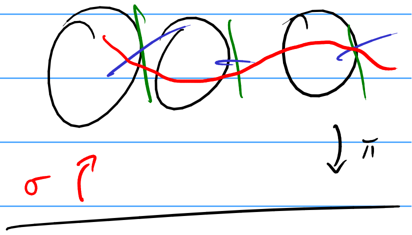
$$\begin{tikzcd} F \arrow[r, hook] & E \arrow[d, "\pi"] \\ & X \end{tikzcd}$$



get an exact sequence

$$0 \to T_\pi \longrightarrow TE \xrightarrow{D\pi} \pi^* TX \to 0$$

↪ vertical tangent space
for each $\pi^{-1}(\text{point}) \cong F$,
$T_\pi|_F = T_F$



a connection on E is a splitting of this exact seq

aka a choice of "horizontal" subbundle of TE.

Remark: if F is a discrete space (if π is a covering) then we get a connection for free.

given a section $s: X \longrightarrow E$ can take its derivative $Ds: TX \longrightarrow s^*TE$

pull back the splitting, get

$$0 \to s^*T_\pi \rightleftarrows s^*TE \longrightarrow s^*\cancel{\pi^*}TX \to 0$$

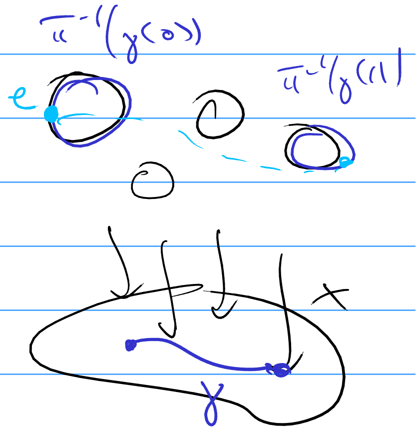
compose with Ds, get a map $TX \to s^*T_\pi$

cov. deriv. of s is a 1-form with values is s^*T_π

Concretely: for $x \in X$, $s^*T_\pi\big|_x = T(\underbrace{\pi^{-1}(x)}_{\cong F})\Big|_{s(x)}$

parallel transport: given a path $\gamma: [0,1] \longrightarrow X$ and a point $e \in \pi^{-1}(\gamma(0))$

$\exists!$ lift $\tilde{\gamma}: [0,1] \longrightarrow E$ s.t. $\nabla_{\gamma'(t)} \tilde{\gamma}(t) = 0$



determines an iso

$$\pi^{-1}(\gamma(0)) \cong F \longrightarrow \pi^{-1}(\gamma(1)) \cong F$$

Earlier: if E is a (real or complex) vector bundle then $\mathbb{R}^*$ or $\mathbb{C}^*$ acts on E by rescaling fibers

require that the splitting is inv. under this action then we get a conn. in our earlier sense.

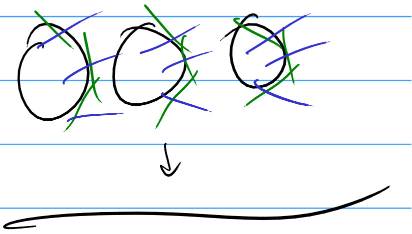
BTW, $T_\pi = \pi^*E$ $\quad s^*T_\pi = E$

Now, if $E = P$ is a principal G-bundle (e.g. frame bundle of a vector bundle $G = GL_n$ or the orth frame bundle of a Riem. v.b.) $G = O(n)$

$$\begin{tikzcd} G \arrow[r, hook] & P \arrow[d, "\pi"] \\ & X \end{tikzcd}$$

G acts on P on the right, freely + trans. on each fiber,

should require the splitting to be invariant under this action



notice: $T_1G =: \mathfrak{g}$ $\qquad$ for $G = GL_n$, $\mathfrak{g} = \mathfrak{gl}_n$ = all $n \times n$ matrices

right action of G on G identifies $T_xG = T_1G = \mathfrak{g}$ $\qquad$ for $G = SO(n)$ $\quad A^TA = 1$

$\mathfrak{g} = \mathfrak{so}(n)$ $\quad A^T + A = 0$

so $TG \cong \mathcal{O}_G \otimes \mathfrak{g}$ trivial bundle modeled on $\mathfrak{g}$.

for our principal bundle, $T_\pi \cong \mathcal{O}_G \otimes \mathfrak{g}$ also trivialized.

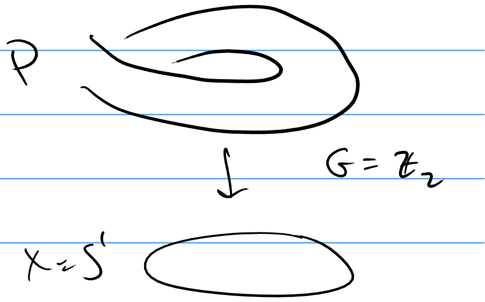
So our splitting $0 \to T_\pi \overset{\longleftarrow}{\longrightarrow} TP \to \pi^*TX \to 0$ can be viewed as a 1-form on P with values in $\mathfrak{g}$.

if G acts on F on the right
build the assoc F-bundle

$$E = P \times F \Big/ (pg, f) \sim (p, gf)$$

A connection on P
induces a connection on E
(worksheet)

A conn. on a v.b.
induces a conn. on the
frame bundle (next time)



$\mathbb{Z}_2 \circlearrowright \mathbb{R}$ by ± 1
assoc bundle is
the Möbius bundle

$\mathbb{Z}_2 \circlearrowright S^1$ by 
assoc bundle is
the Klein bottle