Last Worksheet:

$$\begin{tikzcd} G \arrow[r, hook] & P \arrow[d, "\pi"] \\ & X \end{tikzcd} \qquad G \circlearrowright F$$

$$\begin{tikzcd} E \arrow[d, "\varpi"'] \\ X \end{tikzcd} = P \times_G F := P \times F \Big/ (pg, f) \sim (p, gf)$$

notice: G does not act on E any more.
(maybe the center of G does?)

e.g. if $G = Gl_n$ and $F = \mathbb{R}^n$
then Gl_n doesn't act on E,
only $\mathbb{R}^*$ does

given a connection on P
⇝ get a parallel transport operator

for a piecewise sm. path $\gamma : [0,1] \longrightarrow X$,

get $T : \pi^{-1}(\gamma(0)) \cong G \lhook\joinrel\longrightarrow \pi^{-1}(\gamma(1)) \cong G$

s.t. $T(p) \cdot g = T(p \cdot g)$

⇝ how to get a parallel transport operator on the assoc. bundle E?

$$\varpi^{-1}(x) = \pi^{-1}(x) \times F \Big/ G \qquad (pg, f) \sim (p, gf)$$
$$\text{or } (p, f) \sim (pg, g^{-1} f)$$

Want $T' : \varpi^{-1}(\gamma(0)) \longrightarrow \varpi^{-1}(\gamma(1))$

$$\begin{tikzcd}
\pi^{-1}(\gamma(0)) \times F/\sim \arrow[r] & \pi^{-1}(\gamma(1)) \times F/\sim \\
(p, f) \arrow[r, mapsto] \arrow[d, phantom, "\wr"] & (T(p), f) \arrow[d, bend left] \\
(pg, g^{-1}f) \arrow[r, mapsto] & (T(pg), g^{-1}f)
\end{tikzcd}$$

$$= (T(p)g, g^{-1}(f))$$

⇝ then recover a splitting of $TE \twoheadrightarrow \varpi^* TX$

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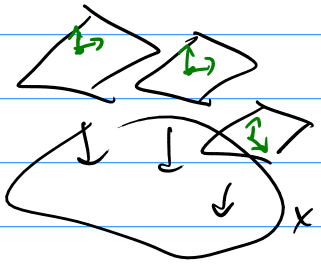
vector bundles ⟷ principal GL_n-bundles

$P \times_{GL_n} \mathbb{R}^n \longleftarrow\!\!| \; P$

connection ⟷ connection

$E \longmapsto$ frame bundle

$\{ x \in X, \text{ basis for } E_x \}$



connection ⟼ connection

parallel transport ⟼ parallel transport

Riemannian metric ⟷ $O(n)$-bundle.

$d(g(s,t)) = g(\nabla s, t) + g(s, \nabla t)$ ⟷ PT will preserve orthonormal frames

so PT preserves g

works for other geometric structures — cx str, G_2 ...

Curvature

given a connection ∇ on a vector bundle E,

define the curvature

for two vector fields $V, W \in \Gamma(TX)$
and a section $s \in \Gamma(E)$

put $F_{V,W}(s) = \nabla_V \nabla_W s - \nabla_W \nabla_V s - \underbrace{\nabla_{[V,W]} s}$

ignore this if $V = \partial/\partial x^i$ and $W = \partial/\partial x^j$

looks like a 2nd order differential op.
$\Gamma(E) \to \Gamma(\Omega^2 \otimes E)$

but it's actually 0th order / a tensor
in $\Gamma(\Omega^2 \otimes E^* \otimes E)$

here's the calc: for $f \in C^\infty(X)$,

want $F_{V,W}(fs) = f \cdot F_{V,W}(s)$

check: $\nabla_V(\nabla_W(fs)) - \nabla_W(\nabla_V(fs)) - \nabla_{[V,W]}(fs)$

$= \nabla_V\big(W(f)\cdot s + f \cdot \nabla_W(s)\big) - \nabla_W\big(V(f)\cdot s + f \cdot \nabla_V(s)\big)$

$- [V,W]f \cdot s + f \nabla_{[V,W]}(s)$

$$= \cancel{V(W(f))\cdot s} + \cancel{W(f)\nabla_V}(s)$$

$$+ \cancel{V(f)\nabla_W}(s) + f\cdot\nabla_V\nabla_W(s)$$

$$- \cancel{W(V(s))} - \cancel{V(f)\nabla_W}(s)$$

$$- \cancel{W(f)\nabla_V}(s) - f\,\nabla_W\nabla_V(s)$$

$$- \left(\cancel{V(W(f))} - \cancel{W(V(f))}\right)\cdot s - f\,\nabla_{[V,W]}(s)$$

$$= f\cdot F_{V,W}(s) \qquad \boxtimes$$

Soon: calculate for round metric on S^2
in stereographic coords.

For a gen. surface in $\mathbb{R}^3$,
compare to Gauss curvature with
osculating circles...

started with ∇ as a map

$$\Gamma(E) \to \Gamma(\Omega^1 \otimes E)$$

can extend to higher differential forms with values in E

$$\begin{tikzcd}
\Gamma(E) \arrow[r, "\nabla"] \arrow[rr, bend right, "\wedge F"'] & \Gamma(\Omega^1 \otimes E) \arrow[r, "\nabla"] \arrow[rr, bend right, "\wedge F"'] & \Gamma(\Omega^2 \otimes E) \arrow[r, "\nabla"] \arrow[rr, bend right, "\wedge F"'] & \Gamma(\Omega^3 \otimes E) \arrow[r] & \cdots
\end{tikzcd}$$

if ∇ is flat, meaning that $F = 0$, this complex computes $H^\bullet$ of the local system of $\mathbb{R}^n$'s

if $E = \mathcal{O}_X$ and $\nabla = d$ then this is the de Rham cx computing $H^\bullet(X, \mathbb{R})$

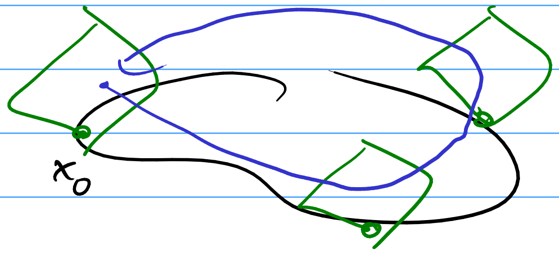
if $X =$ cx mfld, $E =$ holo v.b, and we use ∂

$$\Gamma(E) \to \Gamma(\Omega^{0,1} \otimes E) \to \Gamma(\Omega^{0,2} \otimes E) \to \cdots$$

this is the Dolbeault complex that computes $H^\bullet$(sh. of holo sections of E)

parallel transport for a flat connection determines a hom.

$$\pi_1(X, x_0) \longrightarrow GL_n(\mathbb{R})$$



conversely, such a hom determines a flat conn.

notice: homs. like that have deformations.

next time (or Friday?)

given a conn. on a fiber bundle

$$\begin{tikzcd} 0 \arrow[r] & \underline{T_\pi} \arrow[r] & \underline{TE} \arrow[r] \arrow[l, bend right] & \pi^* TX \arrow[r] \arrow[l, bend right] & \cdot \end{tikzcd}$$

curvature asks whether the Lie bracket of two horz. v.f.'s is still horz