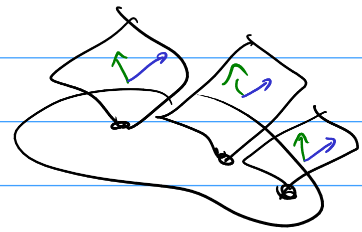
Last time: for a connection ∇ on a vector bundle E,

curvature $F_{v,w}(s) = \nabla_v \nabla_w s - \nabla_w \nabla_v s - \nabla_{[v,w]} s$

Surprise: only depends on the value of s at a point, not in a nbd.

$F \in \Gamma(\Lambda^2 T^* X \otimes \mathrm{End}(E))$



Seen: if E carries a Riem. metric g and $s_1, \ldots, s_n$ is an orthonormal triv. (over $U \subset X$)

and we write ∇ in this trivialization

$\nabla s_i = \sum_j \omega_i^{\,j} s_j$ where $\omega_i^{\,j}$ are 1-forms

then ∇ is compatible with g
iff $\omega_i^{\,j} = -\omega_j^{\,i}$
in other words, the 1-form ω takes vals in $\mathfrak{so}(n) \subset \mathfrak{gl}_n \mathbb{R}$

curvature form will also be skew-symm in that case

if $F(s_i) = \sum_j \Omega_i^{\,j} s_j$ where $\Omega_i^{\,j}$ are 2-forms

then $\Omega_j^{\,i} = -\Omega_i^{\,j}$

$\nabla_V \nabla_W s - \nabla_W \nabla_V s - \nabla_{[V,W]} s$

proof: $\nabla_W s_i = \sum_j \omega_i^{\,j}(W)\, s_j$

so $\nabla_V \nabla_W s_i = \sum_j V\left(\omega_i^{\,j}(W)\right) s_j + \sum_{j,k} \omega_i^{\,j}(W)\, \omega_j^{\,k}(V)\, s_k$

$= \sum_j V\left(\omega_i^{\,j}(W)\right) s_j + \sum_{j,k} \omega_i^{\,k}(W)\, \omega_k^{\,j}(V)\, s_j$

$\Omega_i^{\,j}(V,W) = V\left(\omega_i^{\,j}(W)\right) + \sum_k \omega_i^{\,k}(W)\, \omega_k^{\,j}(V)$ switching i and j switches these

$- W\left(\omega_i^{\,j}(V)\right) - \sum_k \omega_i^{\,k}(V)\, \omega_k^{\,j}(W)$

$- \omega_i^{\,j}([V,W])$

$= d\omega_i^{\,j}(V,W) - \frac{1}{2}[\omega,\omega]_i^{\,j}$

$\hookrightarrow$ or $\omega \wedge \omega$

confusing because ω takes values in $\mathfrak{gl}_n\mathbb{R}$ or $\mathfrak{so}(n)$

if ω was skew-symm. in i, j then Ω is too. $\boxed{E}$

Globally, $F \in \Gamma(\Lambda^2 T^*X \otimes \mathfrak{so}(E))$ where $\mathfrak{so}(E) \subset \mathrm{End}(E)$

if ∇ was compat with a cx str. J
meaning that $\nabla(Js) = J(\nabla s)$
so ω takes values in $\mathfrak{gl}_n\mathbb{C} \subset \mathfrak{gl}_{2n}\mathbb{R}$
then same for F.

$F \in \Gamma(\Lambda^2 T^*X \otimes \mathrm{End}_{\mathbb{C}} E)$

looking ahead, F is a 2-form with values in $\mathfrak{so}(E)$ or $\mathrm{End}_{\mathbb{C}}(E)$

take trace, det, other coeffs of char. polynomial of the matrices

$\rightsquigarrow$ get out $2k$-forms with values in $\mathbb{R}$ or $\mathbb{C}$ for various k.

check: they're closed, globally OK
changing the connection ∇ changes this by an exact form
$\rightsquigarrow$ get well-defined class in
$H^{2k}(X, \mathbb{R}$ or $\mathbb{C})$

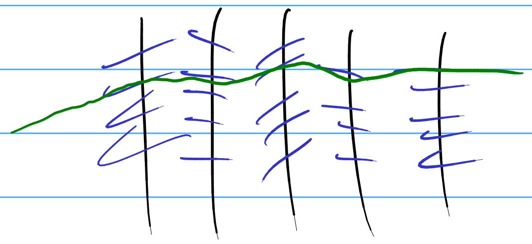
Pontryagin or Chern classes.

Pfaffian $\rightsquigarrow$ Euler class

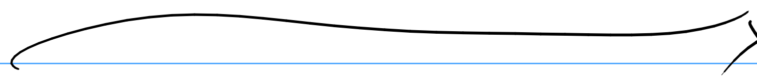
On more gen fiber bundles?
ease into it!

a section s of a v.b. E is
parallel if $\nabla s = 0$ (1-form with vals in E)

in terms of the splitting of
$0 \to p^* E \to TE \to p^* TX \to 0$,



E



X

TE = vertical $\oplus$ horizontal

and the graph of s is tangent to
the horizontal bit.

if we can get $r = \operatorname{rank}(E)$
lin. indep. (local) parallel sections then

∇ is (locally) just the trivial conn. on $\mathcal{O}_X^r$

$$\nabla(f_1, \dots, f_r) = (df_1, \dots, df_r)$$

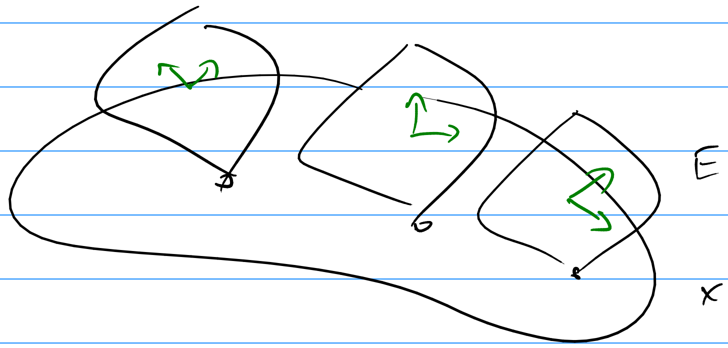
this is
the triv. conn.
on $\mathcal{O}_X^r$

$$\nabla(f_1 s_1 + \dots + f_r s_r) = \sum df_i \cdot s_i + \cancel{f_i \nabla s_i}$$

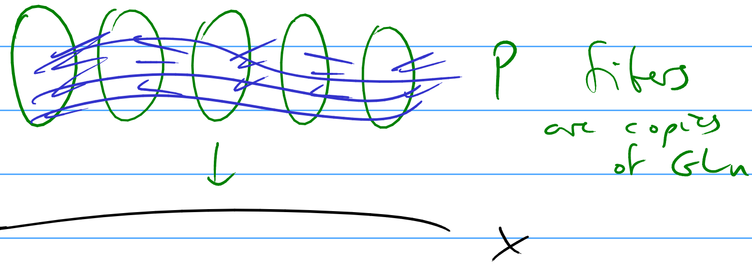
Conversely, if our ∇ is locally tr. V
then (locally) we can get
r lin. indep. parallel sections.

think about the frame bundle

$$\begin{array}{c} P \\ \pi \downarrow \\ X \end{array}$$



a connection ∇ on E
gives a connection on P,



i.e. a splitting
$$TP = \underset{\substack{\wr\wr \\ P \times \mathfrak{gl}_n}}{\text{vert}} \oplus \underset{\substack{\wr\wr \\ \pi^* TX}}{\text{horz}}$$
invariant under left action of GL_n on P

horizontal sub-bundle $\subset TP$
is a "distribution"
Frobenius: comes from a foliation iff
$\forall\ V, W \in \Gamma(\text{horz.}) \subset \Gamma(TP)$

we have $[V, W] \subset \Gamma(\text{horz})$ still

but r lin. indep., parallel local sections of E
give a local section of P thats tangent to the horz. sub-bundle
this + GL_n-equivariance also gives a foliation

__Def__: curvature of a principal bundle
takes two vector fields $V, W \in \Gamma(TX)$

$\rightsquigarrow$ get horizontal lifts $\tilde{V}, \tilde{W} \in \Gamma(TP)$

$\rightsquigarrow$ take vertical part of $[\tilde{V}, \tilde{W}] \in C^\infty(P) \otimes \mathfrak{gl}_n$

descends to a 2-form on X with vals in $\mathfrak{gl}_n$?
same as $\nabla_V \nabla_W -$ blah ?

next time.