Last time: for a connection $\nabla$ on a vector bundle $E$,

\[ \text{curvature } F_{\gamma \mu} (s) = \nabla_{\gamma} \nabla_{\mu} s - \nabla_{\mu} \nabla_{\gamma} s - \nabla_{[\gamma \mu]} s \]

\text{surprise: only depends on the value of } s \text{ at a point, not in a nbhd}

\[ F \in \mathcal{F}(\Lambda^2 \tau^* X \otimes \text{End} (E)) \]

\text{seen: if } E \text{ carries a Riem. metric } g \text{ and } s_1, \ldots, s_n \text{ is an orthonormal fr.v. (over } UC X)\]

and we write $\nabla$ in this trivialization

\[ \nabla s_i = \sum_j w_{ij} s_j \text{ where } w_{ij} \text{ are } 1\text{-forms} \]

then $\nabla$ is compatible with $g$ if $w_{ij} = - w_{ji}$

in other words, the 1-form $w$ takes vals in so(n) < gl(n)

curvature form will also be skew-symm in that case

if $F(s_i) = \sum_j \Omega_{ij} s_j$ where $\Omega_{ij}$ are 2-forms

then $\Omega_{ij} = - \Omega_{ji}$
proof: \[ \nabla_w s_i = \sum_j \omega_i^j(w) s_j \]

so \[ \nabla_v \nabla_w s_i = \sum_j v(\omega_i^j(w)) s_j + \sum_{j \neq i} \omega_i^j(w) \omega_j^k(v) s_k \]

= \sum_j v(\omega_i^j(w)) s_j + \sum_{j \neq i} \omega_i^j(w) \omega_j^k(v) s_j

\[ \omega_i^j(v, w) = V(\omega_i^j(w)) + \sum_k \omega_i^k(w) \omega_j^i(u) \]

- \[ W(\omega_i^j(u)) - \sum_k \omega_i^k(u) \omega_j^i(w) \]

- \[ \omega_i^j([u, w]) \]

= \[ \omega_i^j(v, w) - \frac{1}{2} [\omega_i, \omega_j]^i_0 \]

contradict because \( \omega \) takes values

in \( gl_n \mathbb{R} \) or \( so(n) \)

if \( \omega \) was skew-symmetric in \( i, j \)

then \( \omega \) is too.

Globally, \( F \in P(\Lambda^2 TX \otimes so(E)) \) where \( so(E) \subset End(E) \)

if \( v \) was compat with a cx str. \( J \)

meaning that \( \nabla(J s) = J(\nabla s) \)

so \( \omega \) takes values in \( gl_n \mathbb{C} \subset gl_n \mathbb{R} \)

then same for \( F \), \( F \in P(\Lambda^2 TX \otimes End_a E) \)
Looking ahead, $F$ is a 2-form with values in $\text{so}(E)$ or $\text{End}_\mathbb{C}(E)$.

Take trace, det, other coeffs of char. polynomial of the matrices.

We get out 2k-forms with values in $\mathbb{R}$ or $\mathbb{C}$ for various $k$.

Check: they're closed, globally O.K.

Changing the connection $\nabla$ changes this by an exact form.

We get well-defined class in $H^k(X, \mathbb{R} \text{ or } \mathbb{C})$.

Pontryagin or Chern classes.

Pfaffian $\leftrightarrow$ Euler class.
On more gen fiber bundles?

A section $s$ of a v.b. $E$ is parallel if $\nabla s = 0$ (1-form with $s(x)$ in $E$)

In terms of the splitting of

$$0 \rightarrow \pi^* E \rightarrow \pi^* T X \rightarrow 0,$$

$TE = \text{vertical} \oplus \text{horizontal}$

and the graph of $s$ is tangent to the horizontal of $T X$.

if we can get $r = \text{rank} (E)$

indep (local) parallel sections, then

$\nabla$ is (locally) just the trivial conn. on $\mathbb{R}^r$

$$\nabla (f_1, \ldots, f_r) = (df_1, \ldots, df_r)$$

This is the trivial conn. on $\mathbb{R}^r$.

$$\nabla (f_1, \ldots, f_r, s) = \sum df_i \cdot s_i + f_i \cdot \nabla s$$
Concretely, if our $\nabla$ is (locally) $\nabla \nabla$, then (locally) we can get a linearly independent, parallel sections.

Think about the frame bundle $P$.

A connection $\nabla$ on $E$ gives a connection on $P$.

i.e. a splitting $TP = \text{vert} \oplus \text{horz}$

\[ \pi^* TX \oplus \pi^* TX \]

Invariant under left action of $GL_n$ on $P$

Horizontal sub-bundle $\subset TP$ is a "distribution".

Frobenius: comes from a foliation if

$\forall V, W \in \Gamma(\text{horz}) \subset \Gamma(TP)$

we have $[V, W] \in \Gamma(\text{horz})$ still.

But r. lin. indep. global sections of $E$ give a section of $P$ that's tangent to the horz. sub-bundle. This + $GL_n$-equi. also gives a foliation.
Def: curvature of a principal bundle takes two vector fields $\mathbf{v}, \mathbf{w} \in \Gamma(TX)$

$\Rightarrow$ get horizontal lifts $\tilde{\mathbf{v}}, \tilde{\mathbf{w}} \in \Gamma(TP)$

$\Rightarrow$ take vertical part of $(\tilde{\mathbf{v}}, \tilde{\mathbf{w}}) \in C^\infty(P) \otimes \mathfrak{g}_U$ descends to a 2-form on $X$ with vals in $\mathfrak{g}_U$? Same as $\tilde{\mathbf{v}} \mathcal{L}_{\mathbf{w}}$ - blah?

Next time.