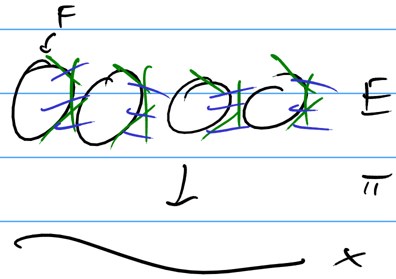
Curvature on Fiber Bundles.

$$\begin{array}{ccc} F & \hookrightarrow & E \\ & & \downarrow \pi \\ & & X \end{array}$$



a connection on E
was a splitting $TE \cong \text{horizontal} \oplus \text{vertical}$

$$= \pi^* TX \oplus T_\pi$$

a section $s: X \to E$ was parallel if
$Ds: TX \to s^* TE \cong TX \oplus s^* T_\pi$
the second component is zero.

curvature 1-form ω on E with values in T_π
just projects $TE \twoheadrightarrow T_\pi$
annihilates the horizontal sub-bundle
identity on vertical sub-bundle

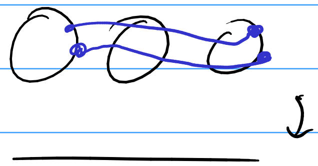
curvature: given two horizontal vector fields
$V, W \in \Gamma(\text{horizontal sub-bundle}) \subset \Gamma(TE)$

ask if $[V, W] \in \Gamma(\text{horz})$ or jumps out.

curvature measures how much it jumps out
$\omega([V, W])$

if curvature $= 0$, then horz. sub-bundle of TE
gives a foliation of E
leaves are transverse to fibers of π

give parallel sections of $E \longrightarrow X$



For a curve $\gamma: [0,1] \longrightarrow X$
get a "parallel transport" map
$\pi^{-1}(\gamma(0)) \cong F \longrightarrow \pi^{-1}(\gamma(1)) \cong F$

if γ is a loop, get a diffeo
$\pi^{-1}(\gamma(0)) \circlearrowleft$ "holonomy"

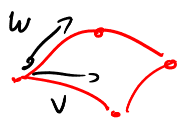
an elt. of $\mathrm{Diffeo}(F)$, well-defined up to conjugation?

if the curvature vanishes,
then parallel transport moves within the leaves of the foliation

if the loop is contractible then
holonomy $= 1$.

only interesting holonomy comes from
non-trivial loops $\gamma \in \pi_1(X, x_0)$ "monodromy"

curvature is measuring the fact that
holonomy around small loops might not be 1.



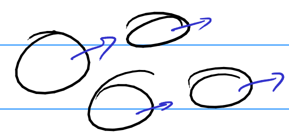
$\nabla_v \nabla_w - \nabla_w \nabla_v$ measures
$\lim_{\square \to .}$ of holonomy.

If $P \xrightarrow{\pi} X$ is a principal bundle,
so G acts on P on the left...

we required that our splitting $TP \cong \pi^* TX \oplus \underset{\substack{\cong \\ \mathcal{O}_P \otimes \mathfrak{g}}}{\underline{T_\pi}}$
be equivariant.

given two vector fields $v, w \in \Gamma(TX)$
$\exists!$ "horizontal lifts" $\tilde{v}, \tilde{w} \in \Gamma(\text{horz}) \subset \Gamma(TE)$

$[\tilde{v}, \tilde{w}] \rightsquigarrow$ proj. into vert. sub-bundle
$\rightsquigarrow$ get $\Gamma(T_\pi)$



for a principal bundle that's
$\Gamma(\mathcal{O}_P \otimes \mathfrak{g})$ – function on P
with values in $\mathfrak{g}$.

does it descend to a function on X w/ vals in $\mathfrak{g}$?
is it G-invariant?
$\hookrightarrow$ yes: action of G preserves splitting,
so preserves $\tilde{v}, \tilde{w}, [\tilde{v}, \tilde{w}]$

get a map $\Gamma(TX) \otimes \Gamma(TX) \to C^\infty(X) \otimes \mathfrak{g}$

actually $\Gamma(\Lambda^2 T^*X \otimes \mathfrak{g})$ $\mathfrak{g}$-valued 2-form.

finish this story Monday
worksheet: comparing $\nabla_v \nabla_w - \nabla_w \nabla_v$ to Gauss curvature
Soon: how curvature transforms, Bianchi id
$\rightsquigarrow$ get coho classes from tr, det, etc.