Curvature on Fiber Bundles.

\[ F \to E \]
\[ \downarrow \pi \]
\[ X \]
\[ \pi \]

A connection on \( E \) is a splitting \( \pi^* TX \oplus T_\pi \).

A section \( s: X \to E \) was parallel if \( Ds: TX \to s^* TE \cong TX \oplus s^* T_\pi \) has the second component is zero.

Curvature 1-form \( \omega \) on \( E \) with values in \( T_\pi \) just projects \( TE \to T_\pi \).

Availability of horizontal sub-bundle identifies vertical sub-bundle

Curvature: given two horizontal vector fields \( V, W \in \Gamma (\text{horizontal sub-bundle}) \subset \Gamma (TE) \)

ask if \( [V, W] \in \Gamma (\text{horiz}) \) or jumps out.

Curvature measures how much it jumps out \( \omega ([V, W]) \).
if curvature = 0, then horiz. sub-bundle of TE
gives a foliation of E
leaves are transverse to fibers of π

give parallel sections of $E \to X$

For a curve $γ : [0, 1] \to X$

get a "parallel transport" map
$\pi^{-1}(γ(0)) \cong F \to \pi^{-1}(γ(1)) \cong F$

if $γ$ is a loop, get a diffeo
$\pi^{-1}(γ(0)) \cong \text{"holonomy"}$

in elt. of $\text{Diff}_0(F)$, well-defined
up to conjugation?

if the curvature vanishes,
then parallel transport moves
within the leaves of the foliation

if the loop is contractible then
holonomy = 1.

only interesting holonomy comes from
non-trivial loops $γ \in \pi_1(X, x_0)$ "monodromy"

curvature is measuring the fact that
holonomy around small loops might not be 1.

$\nabla \nabla w - \nabla w \nabla v$ measures
imbalance of holonomy.
If $P \longrightarrow X$ is a principal bundle, so $G$ acts on $P$ on the left...

we required that our splitting $TP = \pi^*TX \oplus T_{\pi}$ be equivariant.

given two vector fields $U, W \in \Gamma(TX)$

$\pi$: "horizontal lifts" $\tilde{U}, \tilde{W} \in \Gamma(\text{horz}) \subset \Gamma(TE)$

$[\tilde{U}, \tilde{W}] = \text{proj into vert. sub-bundle}$

$\rightarrow \Gamma(T_{\pi})$

for a principal bundle that's

$\Gamma(g \circ \phi)$ - function on $P$

with values in $g$.

does it descend to a function on $X$ with values in $g$?

is it $G$-invariant?

$\Rightarrow$ yes: action of $G$ preserves splitting, so preserve $\tilde{U}, \tilde{W}, [\tilde{U}, \tilde{W}]$

get a map $\Gamma(TX) \otimes \Gamma(TX) \rightarrow \mathcal{A}^2(X) \otimes g$

actually $\Gamma(\Lambda^2(X \otimes g))$ $g$-valued 2-form.

finish this story Monday

worksheet: comparing $\nabla_Du - \nabla_DDu$ to Gauss curvature

soon: how curvature transforms, Bianchi id

get coho classes from $tr$, $det$, etc.