For a v.b. $E \to X$ with a connection ∇

if we choose local sections
$e_1, \dots, e_r \in \Gamma(E|_U)$
that give a trivialization of $E|_U$,

we defined the Christoffel symbols

$$\nabla_{\partial_i}(e_j) = \sum_k \Gamma_{ij}^k e_k$$

and the connection 1-form

$$\nabla(e_j) = \sum_k \omega_j^k e_k \qquad \left(\text{so } \omega_j^k = \sum_i \Gamma_{ij}^k dx^i\right.$$

Curvature form: F or Ω or R

$$(\nabla_{\partial_i}\nabla_{\partial_j} - \nabla_{\partial_j}\nabla_{\partial_i})(e_k) = \sum_\ell R_{ijk}^\ell e_\ell$$

$$\Omega_k^\ell = \sum R_{ijk}^\ell dx^i \wedge dx^j$$

$\hookrightarrow$ matrix-valued 2-form.

globally, $\Omega \in \Gamma(\Lambda^2 T^*X \otimes \mathrm{End}(E))$

Cartan's (second) structure eqn.

$$\Omega = d\omega - \frac{1}{2}[\omega, \omega]$$

$$\frac{1}{2}[\omega,\omega](V,W) = [\omega(V), \omega(W)]$$

for two 1-forms α, β with values in a Lie alg. $\mathfrak{g}$,

$$[\alpha, \beta](V, W) = \alpha(V)\beta(W) - \beta(W)\alpha(V)$$

antisymmetric in V and W, but

$$- \alpha(W)\beta(V) + \beta(V)\alpha(W)$$

symmetric in α and β?!

$$= [\alpha(V), \beta(W)] - [\alpha(W), \beta(V)]$$

So

$$[\omega, \omega](V, W) = 2\,\omega(V)\omega(W) - 2\,\omega(W)\omega(V)$$

$$= 2[\omega(V), \omega(W)]$$

outcome: when people say "wedge the forms and bracket the Lie alg. parts",

two anti-symmetric operations yield a symmetric one.

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last time we had a curvature form ω on the frame bundle $P \to X$

ω(vector field on P) = vertical component

$\rightsquigarrow$ takes values in $\mathfrak{gl}_n$ or $\mathfrak{so}_n$

curvature form $\Omega(V, W) = \omega([\text{horz part of } V, \text{horz part of } W])$

check: $\Omega = d\omega \pm \frac{1}{2}[\omega, \omega]$

also: for $v, w \in \Gamma(TX)$,

take horizontal lifts $\tilde{v}, \hat{w} \in \Gamma(TP)$

take $\omega([\tilde{v}, \tilde{w}]) \rightsquigarrow$ section of the vert. bundle

= function on P with values in $\underline{\underline{gl_n}}$

Said that it descends to give Ω downstairs

but Ω downstairs takes values in $End(E)$, not just $\mathcal{O}_x \otimes gl_n$

$$\mathcal{O}_x \otimes gl_n = End(\mathcal{O}_x^{\wedge})$$

resolution: vert bundle $\cong \mathcal{O}_p \otimes gl_n$

but GL_n-action is non-trivial

$$\begin{tikzcd} P \arrow[d, "\pi"] \\ X \end{tikzcd}$$

GL_n acts on gl_n by conjugation

or $O(n)$ acts on so_n by conjugation

$\pi^* E \cong \mathcal{O}_p^{\wedge}$

$\pi^* End(E)$ is trivial: $\pi^* End(E) \cong \mathcal{O}_p \otimes gl_n$

but not GL_n-equivariantly trivial.

in fact $\pi^*(End\ E) =$ vertical bundle.

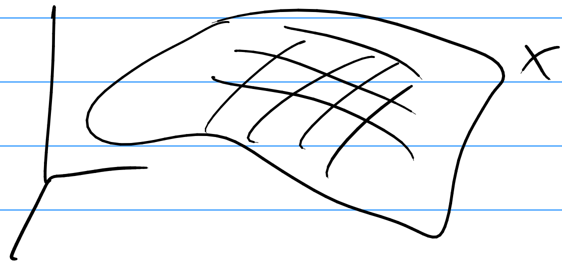
(GL_n equivariantly).

Next time: move toward getting char. classes
from curvature.

Bianchi identity etc.

Worksheet.

$$\phi : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$$



$$\phi(x,y) = \begin{pmatrix} x \\ y \\ f(x,y) \end{pmatrix}$$

∂_x is a section of $T\mathbb{R}^2$

$\phi_*(\partial_x)$ or $D\phi(\partial_x)$ is a section of $\phi^* T\mathbb{R}^3$

aka it's 3 functions on $\mathbb{R}^2$

↪ not $\mathbb{R}^3$
not really X

$$\phi_*(\partial_x) = \begin{pmatrix} 1 \\ 0 \\ f_x \end{pmatrix} \qquad \phi_*(\partial_y) = \begin{pmatrix} 0 \\ 1 \\ f_y \end{pmatrix}$$

standard connection on $T\mathbb{R}^3$ is
just "differentiate the 3 functions"

pulls back to a conn. on $\phi^* T\mathbb{R}^3$
with same description

$f_{xy} = f_{yx}$

$$\nabla_x(\phi_*(\partial_x)) = \begin{pmatrix} 0 \\ 0 \\ f_{xx} \end{pmatrix} \qquad \nabla_y(\phi_* \partial_x) = \begin{pmatrix} 0 \\ 0 \\ f_{xy} \end{pmatrix}$$

$$\nabla_x(\phi_* \partial_y) = \begin{pmatrix} 0 \\ 0 \\ f_{xy} \end{pmatrix} \qquad \nabla_y(\phi_* \partial_y) = \begin{pmatrix} 0 \\ 0 \\ f_{yy} \end{pmatrix}$$

std metric on $\mathbb{R}^3$ is $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} =: g$

pull back via ϕ: get $\begin{pmatrix} 1+f_x^2 & f_x f_y \\ f_x f_y & 1+f_y^2 \end{pmatrix} =: g'$

Levi-Civita connection of g'

$\nabla' = \nabla$ followed by orthogonal projection

$$\phi^* T\mathbb{R}^3 \longrightarrow T\mathbb{R}^2$$

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \longmapsto W := \frac{1}{1+f_x^2+f_y^2}\left(f_x \partial_x + f_y \partial_y\right)$$

So $\nabla'_x(\phi_* \partial_x) = f_{xx} W \qquad \nabla'_y(\phi_* \partial_x) = f_{xy} W$

maybe we should omit this ϕ_*?

$\nabla'_y(\phi_* \partial_x) = f_{xy} W \qquad \nabla'_y(\phi_* \partial_y) = f_{yy} W$

next: compute $\nabla'_x W$ and $\nabla'_y W$, then plug in $(0,0)$ and use $f_x(0,0) = f_y(0,0) = 0$

So $\nabla'_x W = -\dfrac{2 f_x f_{xx} + 2 f_y f_{xy}}{(1+f_x^2+f_y^2)^2} \cdot \left(f_x \partial_x + f_y \partial_y\right)$

$$+ \frac{1}{1+f_x^2+f_y^2}\Big(f_{xx}\partial_x + f_x f_{xx} W + f_{xy}\partial_y + f_x f_{xy} W\Big)$$

evaluate at $(0,0)$ so f_x and f_y drop out

$$\nabla'_x W\Big|_{(0,0)} = 0 + \frac{1}{1}\left(f_{xx}\partial_x + 0 + f_{xy}\partial_y + 0\right)$$

$$= \underline{f_{xx}\partial_x + f_{xy}\partial_y}$$

similarly: $\nabla'_y W = \underline{f_{xy}\partial_x + f_{yy}\partial_y}$

curvature? $\left(\nabla'_x \underline{\nabla'_y} - \nabla'_y \nabla'_x\right)(\underline{\partial_x})\Big|_{(0,0)}$

$$= \nabla'_x(f_{xy}W) - \nabla'_y(f_{xx}W)\Big|_{(0,0)}$$

$$= \cancel{f_{xxy}W} + f_{xy}\left(\cancel{f_{xx}\partial_x} + f_{xy}\partial_y\right)$$

$$- \cancel{f_{xxy}W} - f_{xx}\left(\cancel{f_{xy}\partial_x} + f_{yy}\partial_y\right)$$

$$= \left(\underline{f_{xy}^2 - f_{xx}f_{yy}}\right)\underline{\partial_y}$$

similar: $\left(\nabla'_x \underline{\nabla'_y} - \nabla'_y \nabla'_x\right)(\partial_y) = \left(f_{xx}f_{yy} - f_{xy}^2\right)\partial_x$

curvature form is $\begin{pmatrix} 0 & f_{xx}f_{yy}-f_{xy}^2 \\ f_{xy}^2-f_{xx}f_{yy} & 0 \end{pmatrix} dx\wedge dy$

(at $(0,0)$. notice that

$$dA = \frac{1}{1+f_x^2+f_y^2}\, dx\wedge dy$$

$$dA\big|_{(0,0)} = dx\wedge dy$$