If X = Riem. manifold,
Levi-Civita connection on TX
unique tors. free conn. compat with g.
given by some formula with 3 or 6 terms

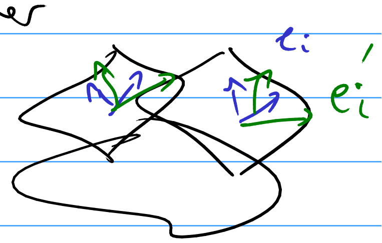
equivalent to a splitting TP = horz $\oplus$ vert
where P = orthonormal frame bundle
(principal $O(n)$ bundle)
↳ does this splitting have a nice description?
or maybe L-C conn. on T^*X is better?

E = real or cx vector bundle on X
∇ = connection on E

take a local triv. $e_1, \ldots, e_n \in \Gamma(E|_U)$

get connection form $\omega_i^{\ j}$ = matrix of 1-forms
curvature form $\Omega_i^{\ j} = d\omega + \frac{1}{2}[\omega, \omega]$
matrix of 2-forms.

Lemma: suppose $e_1', \ldots, e_n'$ is another
local trivialization.
write $\underline{e_i'} = \sum a_i^{\ j} e_j$



$$(\nabla_x \nabla_y - \nabla_y \nabla_x) e_i = \sum_j \Omega_i^{\ j}(\partial_x, \partial_y) e_j$$

$$(\nabla_x \nabla_y - \nabla_y \nabla_x) e_i' = \sum_j \Omega_i'^{\ j}(\partial_x, \partial_y) e_j'$$

how is Ω related to Ω'?

$$(\nabla_x\nabla_y - \nabla_y\nabla_x)e_i' = \sum_k (\nabla_x\nabla_y - \nabla_y\nabla_x)a_i^k e_j$$

$$= \sum_k a_i^k (\nabla_x\nabla_y - \nabla_y\nabla_x) e_k$$

$$= \sum_{k,l} a_i^k \Omega_k^l(\partial_x, \partial_y) e_l$$

$$= \sum_{k,l,j} a_i^k \Omega_k^l(\partial_x, \partial_y)(a^{-1})_l^j e_j'$$

so $\Omega_i'^j = \sum_{k,l} a_i^k \Omega_k^l (a^{-1})_k^j$

more matrixy: $\Omega' = A\Omega A^{-1}$ where $A = (a_i^j)$

ω' vs ω is more complicated.
$\omega' = A\omega A^{-1} \pm A^{-1}dA$?
get it right later
when we need it.

matrizes of 1-forms are slightly scary
as we saw last time. $[\alpha, \beta]$ ...

but 2 forms commute, so
matrizes of them are not scary.

any function $\{\text{matrices}\} \longrightarrow \{\text{numbers}\}$ that's conjugation invariant

will take my (locally defined) curvature form Ω

and give something globally defined.

suppose M is an $n \times n$ matrix with eigenvalues $\lambda_1, \dots, \lambda_n$

the char. poly.

$$\det(tI - M) = (t-\lambda_1)(t-\lambda_2) \cdots (t-\lambda_n)$$

$$\begin{aligned}
= \; & t^n \\
& - t^{n-1}(\underbrace{\lambda_1 + \lambda_2 + \dots + \lambda_n}_{tr(M)}) \\
& + t^{n-2}(\underbrace{\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 + \dots}) \\
& + \dots \\
& \pm \underbrace{\lambda_1\lambda_2 \cdots \lambda_n}_{\det(M)}
\end{aligned}$$

elem. symm. functions of λ_i

middle ones can be described as

$$tr(\Lambda^k M : \Lambda^k \mathbb{R}^n \longrightarrow \Lambda^k \mathbb{R}^n)$$

all conj. invariant:

$$\det(t \cdot I - AMA^{-1})$$

$$= \det(A(tI - M)A^{-1})$$

$$= \cancel{\det A} \; \det(tI - M) \; \cancel{\det A^{-1}}$$

for a $\mathbb{C}$ vector bundle, define the Chern classes:

$c_1(\nabla) = tr(\Omega)$ a $\mathbb{C}$-valued 2-form

$c_2(\nabla) =$ second thing (Ω) a $\mathbb{C}$-valued 4-form

$c_n(\nabla) = \det(\Omega)$ a $2n$-form.

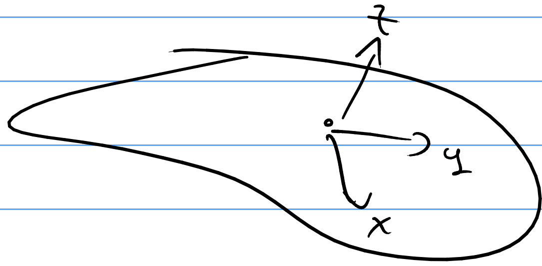
globally defined.

for a real v.b. with Riem. metric,
Ω is a skew matrix

so $tr(\Omega) = 0$
in fact all the odd ones vanish
even ones $\rightsquigarrow$ Pontryagin classes
$Pf = \sqrt{\det}$ $\rightsquigarrow$ Euler class.

Soon: these forms are closed.
change the connection $\rightsquigarrow$ change these by an exact form.

Worksheet:



$X \subset \mathbb{R}^3$

X is loc. the graph of a function with $f_x(0,0) = f_y(0,0) = 0$.

studied curvature of LC connection. finish the WS... Gauss curvature.