Last time: given a connection ∇ on a vector bundle E

$\rightsquigarrow$ curvature form $\Omega_\nabla \in \Gamma(\Lambda^2 T^*X \otimes \mathrm{End}(E))$

$\rightsquigarrow$ Chern forms

$c_k(\nabla)$ = k^{th} coeff of the char. poly. of Ω $\in \Gamma(\Lambda^{2k} T^*X)$

$c_1 = \mathrm{tr}(\Omega)$

"Chern-Weil theory"

$c_{top} = \det(\Omega)$

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coeffs of char. poly are the elem. symm functions of eigenvalues

but there are other interesting symmetric functions of eigenvalues

e.g. $\mathrm{tr}(M^k) = \lambda_1^k + \dots + \lambda_n^k$

Chern character $\mathrm{ch}(\nabla) = \mathrm{tr}(\exp(\Omega))$

$$\mathrm{ch}_k(\nabla) = \frac{1}{k!}\mathrm{tr}(\underbrace{\Omega \wedge \dots \wedge \Omega}_{k \text{ times}}) \in \Gamma(\Lambda^{2k} T^*X)$$

related to Chern classes via Newton's identities

$\mathrm{ch}_1 = c_1$ $\quad$ $\mathrm{ch}_2 = \frac{1}{2}c_1^2 - c_2$

because $\frac{1}{2}(x^2+y^2) = \frac{1}{2}(x+y)^2 - xy$

$$ch_3 = \frac{1}{6} c_1^3 - \frac{1}{2} c_1 c_2 + \frac{1}{2} c_3$$

Want: c_i are closed. $\to$ now

if ∇' is another connection

then $c_i(\nabla) - c_i(\nabla')$ is exact $\to$ worksheet

$\rightsquigarrow$ get well-defined class in

$H^{2i}_{dR}(X, \mathbb{C})$

depending only on E, not ∇.

Claim: $c_k(\nabla)$ is closed $\forall k$

Enough to prove that $tr(\Omega^k) = k!\, ch_k(\nabla)$ is closed

$$d\, tr(\Omega^k) = tr(d\Omega \wedge \Omega \wedge \dots \wedge \Omega) + tr(\Omega \wedge d\Omega \wedge \Omega \wedge \dots)$$

$+ \; k-2$ more

$$= k\, tr(d\Omega \wedge \Omega^{k-1})$$

Need the 2^{nd} or differential Bianchi identity:

$$d\Omega = \Omega \wedge \omega - \omega \wedge \Omega$$

Then $tr(d\Omega \wedge \Omega^{k-1}) = tr(\Omega \wedge \omega \wedge \Omega^{k-1})$

$$- tr(\omega \wedge \Omega \wedge \Omega^{k-1}) = 0$$

cancel because

$tr(AB) = tr(BA)$

Pf of Bianchi identity:

start from Cartan structure eq:

$$\Omega_i^{\ j} = d\omega_i^{\ j} - \sum_k \omega_i^{\ k} \wedge \omega_k^{\ j}$$

$$d\Omega_i^{\ j} = 0 - \sum_k d\omega_i^{\ k} \wedge \omega_k^{\ j} + \sum_k \omega_i^{\ k} \wedge d\omega_k^{\ j}$$

$$= -\sum_{k,l} \left(\Omega_i^{\ k} + \cancel{\omega_i^{\ l} \wedge \omega_l^{\ k}}\right) \wedge \omega_k^{\ j}$$

$$+ \sum_{k,l} \omega_i^{\ k} \wedge \left(\Omega_k^{\ j} + \cancel{\omega_k^{\ l} \wedge \omega_l^{\ j}}\right)$$

$$= -\omega \wedge \Omega + \Omega \wedge \omega$$

(if you think carefully about how matrix mult. looks entry-wise)

worksheet: $c_k(\nabla') - c_k(\nabla)$ is exact.

no indices!