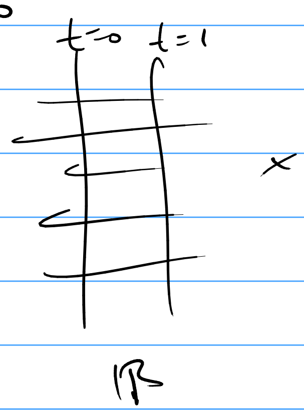
Last worksheet: if α is a closed form on $X \times \mathbb{R}$, let $\alpha_t = \alpha|_{X \times \{t\}}$

then $\alpha_1 - \alpha_0$ is exact.

Secretly: pullback on de Rham coho is homotopy invariant



if $X =$ surface, $\alpha = 1$ form do it by hand to get the flavor.

$$\alpha = f(x,y,t)\,dx + g(x,y,t)\,dy + h(x,y,t)\,dt$$
$$\alpha_0 = f(x,y,0)\,dx + g(x,y,0)\,dy$$
$$\alpha_1 = \text{sim.}$$

$$d\alpha = (g_x - f_y)\,dx \wedge dy + \overset{0}{\overset{\shortparallel}{(f_t - h_x)}}\,dx \wedge dt + (g_t - h_y)\,dy \wedge dt$$

$$\alpha_1 - \alpha_0 = \left(\int_0^1 f_t(x,y,t)\,dt\right) dx + \left(\int_0^1 g_t(x,y,t)\,dt\right)$$

$$= \left(\int_0^1 h_x(x,y,t)\,dt\right) dx + \left(\int_0^1 h_y(x,y,t)\,dt\right)$$

put $\beta = \int_0^1 h(x,y,t)\,dt$ $\rightsquigarrow$ 0-form on X

$d\beta = \alpha_0 - \alpha_1$ by differentiating under $\int$.

in general, use Cartan's magic formula:

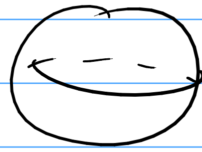
$$\mathcal{L}_{\partial_t} \alpha = \cancel{\frac{\partial}{\partial t} \lrcorner\, d\alpha} + d\left(\frac{\partial}{\partial t} \lrcorner\, \alpha\right)$$

Read Lee...

Said that $c_k(\nabla)$ give well-defined classes in
$H^{2k}(X, \mathbb{C})$

normalization issue: should have taken
$tr(\frac{i}{2\pi}\Omega)$ and so on

example: round metric of radius r
on TS^2



Gauss curvature $= 1/r^2$

Saw that $\Omega = \begin{pmatrix} 0 & 1/r^2 \\ -1/r^2 & 0 \end{pmatrix} dA$

$a+bi \leftrightarrow \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$

regard TS^2 as a cx line bundle. $SO(2) \cong U(1)$

then $\Omega = -i \cdot \frac{1}{r^2} dA$ cx-valued 2-form.

$\int tr(\Omega) = i \cdot \frac{1}{r^2} \cdot 4\pi r^2 = -4\pi i$

wanted $c_1(E)$ = Poincaré dual to vanishing of
a gen. section,

and a gen. v.f. on S^2
vanishes at 2 points.

So use $\frac{i}{2\pi}\Omega \rightsquigarrow \frac{i}{2\pi} \cdot -4\pi i = 2$

Also on WS, if $f: X \to Y$
and E a vector bundle on Y
with a connection ∇

then the pull-back connection $f^*\nabla$ on f^*E ?

we'll have $\Omega_{f^*\nabla} = f^*\Omega_\nabla$

so $c_k(f^*\nabla) = f^*c_k(\nabla)$ as we hope.

$f^*\nabla$ is determined by:

for a tangent vector $v \in T_x X$
and a section $s \in \Gamma(E)$,

get $f_* v \in T_{f(x)} Y$

and a pull-back section $f^*s \in \Gamma(f^*E)$

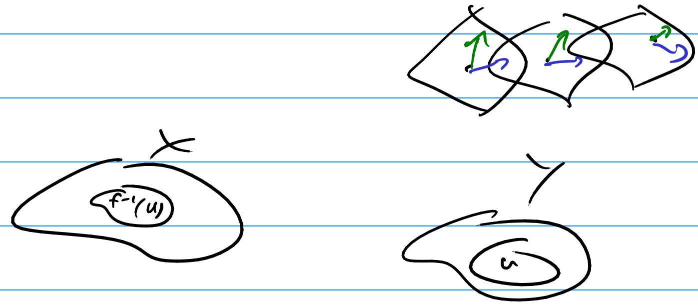
want $(f^*\nabla)_v (f^*s) \in (f^*E)_x$

$$\begin{array}{ccc} (f^*\nabla)_v (f^*s) & \in & (f^*E)_x \\ \| & & \| \\ \nabla_{f_*v}(s) & \in & E_{f(x)} \end{array}$$

$$\begin{tikzcd}
X \arrow[dr, dashed] \arrow[drr, bend left, "s \circ f"] \arrow[ddr, equals] & & \\
& f^*E \arrow[r] \arrow[d] & E \arrow[d] \\
& X \arrow[r, "f"'] & Y \arrow[u, bend right, "s"']
\end{tikzcd}$$

not every section of f^*E is
f^* of a section of E

but locally every section of f^*E
is a $C^\infty(X)$-lin. combo. of pull-back sections
so this determines $f^*\nabla$.



for $f: X \to Y$

get $f^*: C^\infty(Y) \longrightarrow C^\infty(X)$

$g \longmapsto g \circ f$

is it true that

$$\Gamma(f^*E) = \Gamma(E) \otimes_{C^\infty(Y)} C^\infty(X) \ ?$$

I think so...

not true that

$$C^\infty(X \times Y) = C^\infty(X) \otimes_{\mathbb{R}} C^\infty(Y)$$

because $\otimes$ only involves finite sums.

$$\mathbb{R}[x] \otimes_{\mathbb{R}} \mathbb{R}[y] = \mathbb{R}[x,y]$$

but $C^\infty(\mathbb{R}) \otimes_{\mathbb{R}} C^\infty(\mathbb{R}) \neq C^\infty(\mathbb{R}^2)$

$e_1, \dots, e_r \in \Gamma(E|_u)$

trivialization

$\rightsquigarrow f^*e_1, \dots, f^*e_r$

$\in \Gamma(f^*E|_{f^{-1}(u)})$

is a triv.

so a local section of $f^*E|_{f^{-1}(u)}$

can be written as

$$\sum_i g_i f^*e_i$$

$$\nabla = dg_i \otimes f^*e_i + g_i \cdot f^*\nabla e_i$$

Dual bundles, $\oplus$, $\otimes$

Chern classes: if $\operatorname{rank}(E) = r$

$$\det\left(tI - \frac{i}{2\pi}\Omega\right) =: t^r - c_1(\nabla)t^{r-1} + c_2(\nabla)t^{r-2} - \cdots$$

so $c_1(\nabla) = \operatorname{tr}\left(\frac{i}{2\pi}\Omega\right)$

total Chern class:

$$c(\nabla) = 1 + c_1(\nabla) + c_2(\nabla) + \cdots$$

$$= \det\left(I + \frac{i}{2\pi}\Omega\right)$$

Chern character

$$ch(\nabla) = r + c_1 + \left(\frac{1}{2}c_1^2 - c_2\right) + (\cdots\cdots)$$

related by Newton's identities

$$= \operatorname{tr}\left(\exp\left(\frac{i}{2\pi}\Omega\right)\right)$$

$$ch_k = \frac{1}{k!}\left(\frac{i}{2\pi}\right)^k \operatorname{tr}(\underbrace{\Omega\wedge\cdots\Omega}_{k \text{ times}})$$

maybe you've seen:

$c_k(E^*) = (-1)^k c_k(E)$ same with ch.

→ $c(E\oplus F) = c(E)\cdot c(F)$

→ $ch(E\oplus F) = ch(E) + ch(F)$

$c(E\otimes F)$ is not nice.

$ch(E\otimes F) = ch(E)\cdot ch(F)$

see it on the level of forms.

given ∇ on E, Ω_∇ is a 2-form with vals in $End(E)$

get a dual connection ∇^* on E^*

Ω_{∇^*} is a 2-form with vals in

$$End(E^*) = End(E)$$
$$A^T \longleftarrow A$$

check: $\Omega_{\nabla^*} = -\Omega_\nabla$

∇^* is determined by:

for $s \in \Gamma(E)$ and $\ell \in \Gamma(E^*)$,

get $\ell(s) \in C^\infty(X)$

then we require $d(\ell(s)) = (\nabla^* \ell)(s) + \ell(\nabla s)$

For $E \oplus F$, have $End(E \oplus F)$

$$= \underline{End(E)} \oplus Hom(E, F) \oplus Hom(F, E) \oplus \underline{End(F)}$$

given ∇ on E and ∇' on F,

$$\Omega_{\nabla \oplus \nabla'} = \Omega_\nabla \oplus \Omega_{\nabla'}$$

$$\begin{array}{c|cc} & E & F \\ \hline E & \Omega_\nabla & 0 \\ F & 0 & \Omega_{\nabla'} \end{array}$$

if A, B are matrices then

$$\det\left(\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right) = \det A \cdot \det B$$

$$\det\left(\begin{array}{c|c} I+\Omega_\nabla & 0 \\ \hline 0 & I+\Omega_{\nabla'} \end{array}\right) = \det(I+\Omega_\nabla)\det(I+\Omega_{\nabla'})$$

so $c(\nabla\oplus\nabla') = c(\nabla)\cdot c(\nabla')$

$$\mathrm{tr}\left(\begin{array}{c|c} \exp\Omega_\nabla & \\ \hline & \exp\Omega_{\nabla'} \end{array}\right) = \mathrm{tr}\exp\Omega_\nabla + \mathrm{tr}\exp\Omega_{\nabla'}$$

so $ch(\nabla\oplus\nabla') = ch(\nabla) + ch(\nabla')$

$\mathrm{End}(E\otimes F) = E^*\otimes F^*\otimes E\otimes F = \mathrm{End}(E)\otimes\mathrm{End}(F)$

connection on $E\otimes F$ is determined by

$$(\nabla\otimes\nabla')(s\otimes t) = \nabla s\otimes t + s\otimes\nabla' t$$

$s\in\Gamma(E)$, $t\in\Gamma(F)$

$$\Omega_{\nabla\otimes\nabla'} = \Omega_\nabla\otimes 1_F + 1_E\otimes\Omega_{\nabla'}$$

$$\mathrm{tr}(\exp(\Omega_{\nabla\otimes\nabla'})) = \mathrm{tr}(\exp(\Omega_\nabla)\otimes\exp(\Omega_{\nabla'}))$$

$$= \mathrm{tr}(\exp(\Omega_\nabla))\cdot\mathrm{tr}(\exp(\Omega_{\nabla'}))$$

$$\mathrm{tr}(A \otimes B) = \mathrm{tr}(A) \cdot \mathrm{tr}(B)$$

(think about diag. entries of $A \otimes B$)

$$\mathrm{ch}(\nabla \otimes \nabla') = \mathrm{ch}(\nabla) \cdot \mathrm{ch}(\nabla')$$