Hopf fibration + Fubini-Study metric

Recall the tautological l.b. on $\mathbb{CP}^n$:

$$\mathcal{O}(-1) = \{ v \in \mathbb{C}^{n+1}, \ell \in \mathbb{CP}^n \mid v \in \ell \} \longrightarrow \mathbb{CP}^n$$
$$(v, \ell) \longmapsto \ell$$

fiber over point $\ell \in \mathbb{CP}^n$
is the 1-dim'l subspace $\ell \subset \mathbb{C}^{n+1}$

$\mathcal{O}(1) = \mathcal{O}(-1)^*$ fiber is ℓ^*

a linear form like $x_0 + 2x_1 - 5x_2 + \cdots$ on $\mathbb{C}^{n+1}$
$\rightsquigarrow$ a section of $\mathcal{O}(1)$

cuts out a hyperplane $H \cong \mathbb{CP}^{n-1} \subset \mathbb{CP}^n$

claim: $c_1(\mathcal{O}(1))$ is Poincaré dual to H:

for a closed $2n-2$ form α on $\mathbb{CP}^n$,

$$\int_{\mathbb{CP}^n} \alpha \wedge c_1(\mathcal{O}(1)) = \int_H \alpha|_H$$

today, just prove $\int_{\mathbb{CP}^n} c_1^n = 1$

if you know $H^*(\mathbb{CP}^n)$ that's enough...

$\mathcal{O}(-1) \subset \underset{\sim}{\mathcal{O}}^{n+1}$ bec. $l \subset \mathbb{C}^{n+1}$

trivial rank $(n+1)$ vector bundle on $\mathbb{CP}^n$

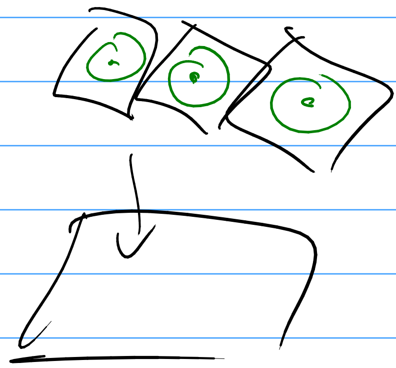
std metric on $\mathcal{O}^{n+1} \rightsquigarrow$ nice one on $\mathcal{O}(-1)$

compute curvature

take the unit circle bundle of $\mathcal{O}(-1)$

it's the principal $U(1)$-bundle

assoc. to $\mathcal{O}(-1)$



in fact it's $S^{2n+1} \subset \mathbb{C}^{n+1}$

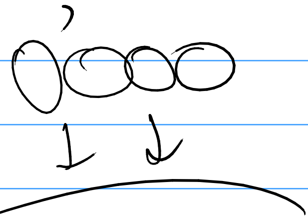
circle $= U(1)$ acts by $u \cdot (z_0, \dots, z_n)$

$= (uz_0, \dots, uz_n)$

in real coords, spins them in pairs

for $(x_0, y_0, x_1, y_1, \dots) \in S^{2n+1}$

spin



$\pi : S^{2n+1} \to \mathbb{CP}^n$

if $n=1$, this is the Hopf fibration

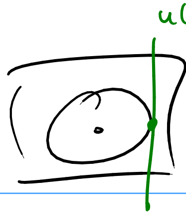
$S^3 \to S^2$

std metric on $\mathbb{C}^{n+1}$ or $\mathbb{R}^{2n+2}$

gives a splitting

$$0 \to \text{vert} \to TS^{2n+1} \to \pi^* T\mathbb{CP}^n \to 0$$

circle $= U(1) = \{ u \in GL_1(\mathbb{C}) \mid \bar{u} u = 1 \}$



<u>vertical bundle</u> (sometimes T_π)

$\cong \underline{\mathcal{O}_S} \otimes u(1)$

$u(1) = i\mathbb{R} \subset gl_1(\mathbb{C})$

$\exp : u(1) \to U(1)$

$it \mapsto e^{it}$

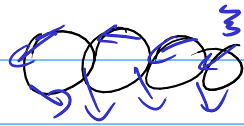
Constant function i in $\Gamma(\mathcal{O}_S \otimes u(1))$

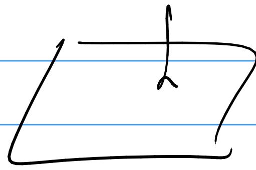
corresp the v.f. that spins the fibers of π

$$\xi = -y_0 \frac{\partial}{\partial x_0} + x_0 \frac{\partial}{\partial y_0} - y_1 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial y_1} + \cdots \in \Gamma(\text{vertical bundle})$$

Connection form

ω : 1-form on S^{2n+1} w/ values in $u(1) = i\mathbb{R}$





$$\omega(V) = g(V, \xi) \cdot i$$

$$g(\xi, \xi) = y_0^2 + x_0^2 + \cdots = 1$$

so $\omega(\xi) = i$

$\omega(\text{horizontal}) = 0$

$$\omega = (\underbrace{-y_0\, dx_0 + x_0\, dy_0 + n \text{ more}}_{\text{"}d\theta\text{"}})\, i$$

$$\int_{\text{fiber circle}} \omega = 2\pi \cdot i$$

Curvature:

$$\Omega = d\omega + \cancel{[\omega, \omega]} \to 0$$

because $u(1)$ is abelian

$$= (2\, dx_0 \wedge dy_0 + 2\, dx_1 \wedge dy_1 + \cdots)\, i$$

invariant under $U(1)$ action?

$$\mathcal{L}_\xi \omega = \cancel{\xi \lrcorner d\omega}^{\,0} + d(\cancel{\xi \lrcorner \omega})^{\,0}$$

if this vanishes then ω and Ω are $U(1)$-inv.

$$\xi \lrcorner \omega = \omega(\xi) = g(\xi,\xi)\cdot i = 1\cdot i$$

$$\xi \lrcorner d\omega = -2(y_0\,dy_0 + x_0\,dx_0 + y_1\,dy_1 + x_1\,dx_1 \cdots)\cdot i$$

$$= -d(y_0^2 + x_0^2 + y_1^2 + x_1^2 + \cdots)\cdot i$$

$$= -d(1)\cdot i = 0$$

$$c_1(\mathcal{O}(-1)) = \frac{i}{2\pi}\Omega = \frac{i}{2\pi}\cdot 2i\,(dx_0 \wedge dy_0 + n \text{ more})$$

$$= -\frac{1}{\pi}(dx_0 \wedge dy_0 + n \text{ more})$$

$$c_1(\mathcal{O}(1)) = \frac{1}{\pi}(dx_0 \wedge dy_0 + n \text{ more})$$

descended to $\mathbb{CP}^n$.

Rmk: pull back to a std. affine patch

$$(1, 0, \mathrm{Re}\, z_1, \mathrm{Im}\, z_1, \dots)$$

$$\begin{tikzcd} & S^{2n+1} \arrow[d, "\pi"] \\ \mathbb{C}^n \arrow[ur] \arrow[r] & \mathbb{CP}^n \end{tikzcd}$$

$$z_1 \cdots z_n \longmapsto (1 : z_1 : \cdots : z_n)$$

get the usual Kähler form of the Fubini-Study metric

$$\frac{i}{2\pi}\partial\bar\partial \log\left(1+|z_1|^2+|z_2|^2+\cdots\right)$$

$=$

$c_1(\mathcal{O}(1))$ on $\mathbb{C}P^n$

want: $\int c_1^n = 1$

$$\begin{array}{c} S^{2n+1} \\ \downarrow \pi \\ \mathbb{C}P^n \end{array}$$

$\omega = i\cdot(-y_0\,dx_0 + x_0\,dy_0 + \cdots)$

$d\omega = 2i(dx_0\wedge dy_0 + \cdots)$

$\pi^* c_1(\mathcal{O}(1)) = -\frac{i}{2\pi}\,d\omega$

$\pi_*\omega = 2\pi i$

so $\int_{\mathbb{C}P^n} c_1^n = \frac{1}{2\pi i}\int_{\mathbb{C}P^n} c_1^n\cdot\pi_*\omega$

$= \frac{1}{2\pi i}\int_{S^{2n+1}} (\pi^* c_1)^n\wedge\omega$

$= \frac{1}{2\pi i}\left(\frac{-i}{2\pi}\right)^n\int_{S^{2n+1}} (d\omega)^n\wedge\omega$

volume form on S^{2n+1}?

ρ = radial vector field

$= x_0\frac{\partial}{\partial x_0} + y_0\frac{\partial}{\partial y_0} + \cdots$

$dV = \rho\,\lrcorner\,(dx_0\wedge dy_0\wedge dx_1\wedge dy_1\cdots)$

$= \rho\,\lrcorner\,\frac{(d\omega)^{n+1}}{(n+1)!\,(2i)^{n+1}}$

$$= \frac{(n+1)\,(p\lrcorner d\omega)\,(d\omega)^n}{(n+1)!\quad (2i)^{n+1}}$$

$$p \lrcorner d\omega = 2\omega$$

$$dVol = \frac{2\underbrace{\omega \;\; n\,(d\omega)^n}}{n!\;\;(2i)^{n+1}}$$

$$\frac{1}{2\pi \cancel{i}}\left(\frac{\cancel{-i}}{\cancel{2\pi}}\right)^n \frac{(\cancel{2}\cancel{i})^{n\cancel{+1}}\; n!}{\cancel{2}} \cdot \int_{S^{2n+1}} dVol$$

$$\|$$

$$\frac{n!}{2\pi^{n+1}} \cdot \text{vol. of } S^{2n+1}$$

$$\|$$

$$1.$$