Last time: on $\mathbb{CP}^n$.

$$\int c_1(\mathcal{O}(1))^n = 1$$

via the Hopf fibration $\pi: S^{2n+1} \to \mathbb{CP}^n$ $\quad U(1) \circlearrowright S^{2n+1}$

$\pi$ was a principal $U(1)$-bundle.

$\mathcal{O}(-1)$ comes from taking the trivial line bundle on $S^{2n+1}$, which is $U(1)$-invariant, and having it descend

in general, if $G \circlearrowright$ freely on $X$ and $E$ is a $G$-equivar. vector bundle on $X$, then $E$ descends to $X/G$

if $\nabla$ is a $G$-inv. connection on $E$, then that descends as well
but you have to pick a connection on the principal $G$-bundle $X \to X/G$

curvature of the descended conn. involves curvature of $\nabla$ upstairs + curvature of the $G$-bundle.

meaty example: $\mathcal{A}$ = all irreducible Hermitian-Einstein connections on $E$
$G$ = gauge group $[\Gamma(\mathrm{Aut}(E))]$
$\nabla$ = tautological connection of $\pi_1^* E$ on $X \times \mathcal{A}$
$\mathcal{A}/G$ = moduli space of stable bundles

Also: Fubini-Study Kähler form should have been

$$\frac{i}{2\pi}\partial\bar\partial \log(1+|z|^2)$$

For a holo. vector bundle $E$
on a Kähler manifold $X$
and a Hermitian metric $h$ on $E$,

got the Chern connection $\nabla$ on $E$:

- compat with $h$
- $\nabla^{0,1} = \bar\partial$

curvature $\Omega = \nabla^2$.

$\bar\partial^2 = 0$ so $\Omega^{0,2} = 0$

if we trivialize $\Omega$ with a unitary frame
$s_1, \dots, s_r \in \Gamma(E|_U)$ $\quad h(s_i, s_j) = \delta_{ij}$

then $\Omega$ is Skew-Hermitian:
$\bar\Omega^T = -\Omega$
so $\Omega^{2,0} = 0$

so $\Omega$ is type $(1,1)$

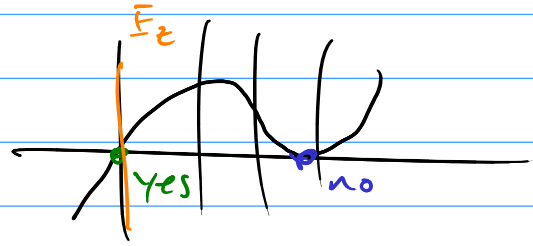
$c_k(\nabla)$ and $ch_k(\nabla)$ are type $(k,k)$

$\rightsquigarrow$ big constraint on which elements of $H^{2k}(X, \mathbb{C})$
can be $c_k$(holo. vector bundle)

cf. Hodge conjecture

# Chern-Gauss-Bonnet

$X$ = sm. manifold of dim $n$
$E$ = cx v.b. of rank $r$
$s \in \Gamma(E)$ cuts out $Z \subset X$ transversely:




at $z \in Z$, get

$$ds: T_z X \to E_z$$

want $ds$ to be surjective.

why not $ds: T_z X$
$\to T_{(z,0)} E$
$\cong \boxed{T_z X} \oplus \boxed{E_z}$
splits because $s(z) = 0$

then $\ker ds = T_z Z$

and $ds$ identifies $N_{Z/X} \cong E|_Z$

(a generic section of $E$ has this property)


Claim: for a closed $(n-r)$-form $\eta$ on $X$, we have

$$\int_X \eta \wedge c_r(E) = \int_Z \eta$$

$\eta$ compactly supp
or $X$ = compact w/o boundary

① indep of choice of connection on $E$

$c_r(\nabla') = c_r(\nabla) + \text{exact}$

$\eta$ is closed so $\int \eta \wedge \text{exact} = 0$

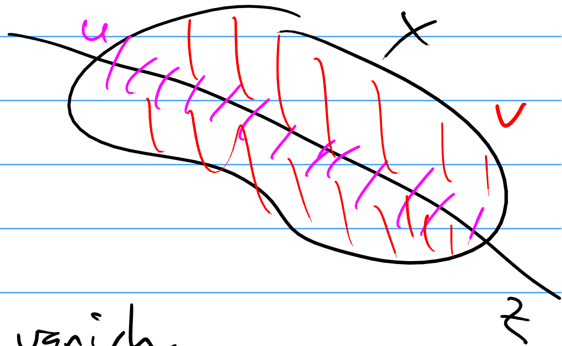
② if $X', E', s', Z'$ are another such package

and $Z' \cong Z$ and $E'|_{Z'} \cong E|_Z$

then claim holds here iff it holds there.

<u>Proof</u>: Let $U \subset X$ be a tubular nbd of $Z$

So $U \cong$ open unit disc bundle in $N_{Z/X} \cong E|_Z$.




Let $V = X \setminus Z$

on $V$, the section $s$ doesn't vanish, so gives a trivial sub-bundle $\mathcal{O}_V \subset E|_V$ (rank 1)

$E|_V = \mathcal{O}_V \oplus E'$

choose a conn. $\nabla_V$ on $E|_V$ that's standard on $\mathcal{O}_V \oplus$ anything on $E'$

then $c_r(\nabla_V) = \cancel{c_1(\mathcal{O}_V \text{ part})}^{0} \cdot c_{r-1}(E' \text{ part})$

aka $\det\left(\begin{array}{c|c} 0 & \\ \hline & ?? \end{array}\right) = 0$

on $\mathcal{U}$, choose any connection $\nabla_u$ on $E|_u$

globally, get a partition of 1 $\varphi$

let $\nabla = \varphi_u \nabla_u + \varphi_v \nabla_v$

$\varphi_u : \mathcal{U} \to [0,1]$
$\varphi_v : V \to [0,1]$
$\varphi_u + \varphi_v = 1$

then $c_r(\nabla)$ is supported in $\mathcal{U}$.

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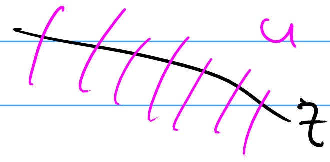
if $X', E', s', Z'$ is another one,

get a tubular nbd $u' \subset X'$
and a connection $\nabla'$ on $E'$
with $c_r(\nabla')$ supp in $u'$




if $Z \cong Z'$ and $E|_Z \cong E'|_{Z'}$

then $u \cong u'$
$E|_u \cong E|_{u'}$ so $c_r(\nabla)$ and $c_r(\nabla')$ differ by an exact form

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going to apply this to $X' = \mathbb{P}(\mathcal{O}_Z \oplus E|_Z)$

take total space of $E|_Z$ - a $\mathbb{C}^r$ bundle
and embed it in a $\mathbb{CP}^r$ bundle
+ recycle our calc. from Friday.

we're encountering the Thom class of
$E|_Z$ ...