Last worksheet: $V = \mathbb{R}^{2n}$

for a skew map $A: V \to V^*$

got $\mathrm{Pf}(A) \in \det V^*$

$\mathrm{Pf}(A)^2 = \det A \in (\det V^*)^{\otimes 2}$

$\mathrm{Pf}(B^T A B) = \det B \ \mathrm{Pf}(A)$

example: $\mathrm{Pf}\begin{pmatrix} 0 & x & y & z \\ -x & 0 & u & v \\ & & 0 & w \\ \text{etc} & & & 0 \end{pmatrix} = xw - yv + uz$

If V has an inner product g, then $V^* \cong V$

so $A: V \to V$ can be skew-symmetric

$$g(Av, w) = -g(v, Aw)$$

now $\det A : \det V \to \det V$

i.e. $\det A$ is a number

if V is oriented, get a std. volume form in $\det V$ or $\det V^*$

$\leadsto$ $\mathrm{Pf}(A)$ is a number too.

reverse orientation $\leadsto$ $\mathrm{Pf}(A)$ becomes $-\mathrm{Pf}(A)$

for $A \in G = SO(2n)$ and $M \in \mathfrak{g} = so(2n) =$ skew-symm matrices

have $$\mathrm{Pf}(A^{-1} M A) = \mathrm{Pf}(A^T M A) = \det(A) \cdot \mathrm{Pf}(M) = 1 \cdot \mathrm{Pf}(M)$$

so if E is an oriented Riem v.b. of rank $2r$
and ∇ is a compat. connection
then $\chi(\nabla) = \mathrm{Pf}\left(\frac{1}{2\pi}\Omega\right)$ is a
globally defined $2r$-form on base.

closed... omitted.
$\nabla \rightsquigarrow \nabla'$ changes χ by an exact form:
same argument as before.

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if E is not oriented then
$\chi(\nabla)$ is a 2-form with values in
$\det E^* \cong \det E$

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if $V = \mathbb{C}^n$ and $A : V \to V$
we know $\det_{\mathbb{R}} A = |\det_{\mathbb{C}} A|^2$

if A is skew-hermitian w.r.t. some
hermitian metric on V (so $\bar{A}^T = -A$)
then the real matrix is skew
$\det_{\mathbb{C}} iA = + \mathrm{Pf}\, A$
could have been $-$, but it's not.

small example: $A = (a + bi)$
$\bar{A}^T = (a - bi)$
so if $\bar{A}^T = -A$ then $A = (bi)$

$$A_{\mathbb{R}} = \begin{pmatrix} 0 & -b \\ b & 0 \end{pmatrix} \qquad \mathrm{Pf} = -b$$

$$\det_{\mathbb{C}} A = bi \qquad \det_{\mathbb{R}} = b^2$$

$A : \mathbb{R}^2 \to \mathbb{R}^2$ let e, f be our basis for $\mathbb{R}^2$

as a 2-form, $A = +b\, e^* \wedge f^*$?

need to resolve this...

$A(e) = bf$
$A(f) = -be$

upshot: $c_r(\text{cx v.b.}) = \chi(\text{underlying real v.b.})$

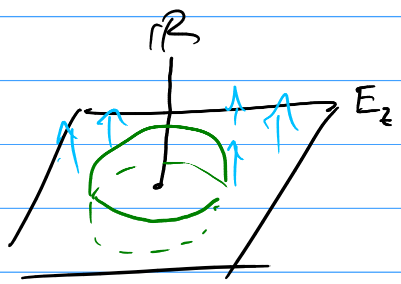
calculation (omitted)
for the round metric on S^{2n},

$$\int \chi(\nabla) = 2$$

now let $E \to Z$ be an oriented Riemannian v.b. of rank $2k$

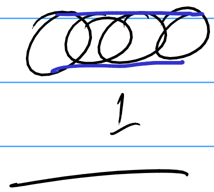
let $X' =$ unit sphere bundle in $E \oplus \mathcal{O}_Z$

get a v.f. on X'
by projecting the vertical v.f. shown here $\longrightarrow$
onto sphere.



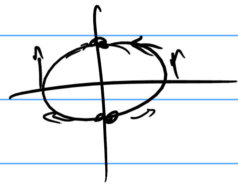
over $z \in Z$

get $s \in \Gamma(TX')$ that vanishes (transversely) at n. and s. pole.

n. and s. pole are sections of $\begin{array}{c} X' \\ \downarrow \\ Z \end{array}$ 

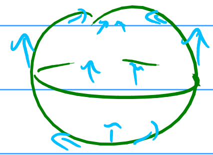
so we identify a nbd of either section in X' with a nbd of zero-section in E.

↳ more explicitly, let t be a coord in the $\mathbb{R}$-direction.
then $\partial/\partial t$ is a v.f. on $E \oplus \mathcal{O}_Z$
sphere bundle $X' \subset E \oplus \mathcal{O}_Z$
+ we have a Riem. metric
so can restrict $\partial/\partial t$ to X'
and proj. orth. into TX'.



if we choose a connection ∇ on TX'
for which s is parallel in a nbd of the equator

then $\chi(\nabla)$ will vanish on that nbd



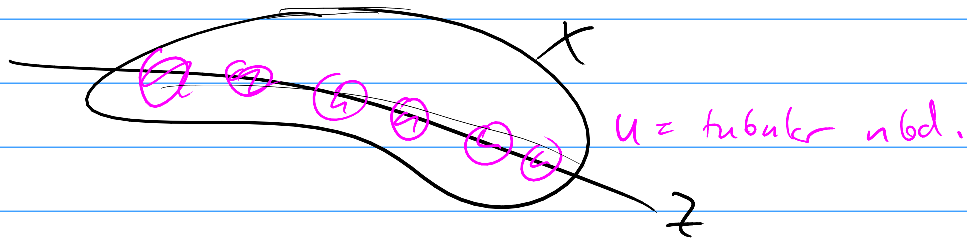
so for α a top-dimensional form on Z,

$$\int_{X'} \chi(\nabla) \cdot \pi^* \alpha = \int_Z \cancel{\pi_* \chi(\nabla)}\, 2 \cdot \alpha$$

$$\|$$

$$\int_{\text{n. hemisphere}} \chi(\nabla) \cdot \pi^* \alpha + \int_{\text{s. hemisphere}} \chi(\nabla) \cdot \pi^* \alpha$$

either of these is the thing we wanted to calculate in our general global situation



rank (E) is even ⇝ orientations of $N_{Z/X'}$ and E agree at both poles.