recall: if $E \to X$ is a $\overset{\mathbb{C}}{\vee}$ v.b. of rank r
and $s \in \Gamma(E)$ cuts out $Z \subset X$ transversely
then $c_r(E)$ is PD to Z.

today: if $s_1, \dots, s_k \in \Gamma(E)$ are generic sections $k \le r$
and $Z \subset X$ is the set where
they're linearly dependent, i.e.

$$Z = \left\{ x \in X \;\middle|\; \begin{array}{l} s_1(x), s_2(x), \dots, s_k(x) \in E_x \\ \text{span a space of} \\ \text{dim} < k \end{array} \right\}$$

then Z is PD to $c_{r-k+1}(E)$

What does "generic" mean?

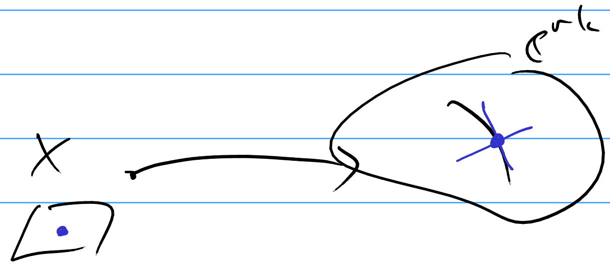
over $U \subset X$, choose a trivialization $E|_U \cong \mathcal{O}_U^r$
then our k sections give a map

$$U \xrightarrow{f} r \times k \text{ matrices} \cong \mathbb{C}^{rk}$$

↳ this is stratified by rank.

want this map to be ☆
transverse to
(each stratum ∖ next)

$$\begin{array}{l} \text{rank} \le k = \mathbb{C}^{rk} \\ \quad \cup \\ \text{rank} \le k-1 \leftarrow \text{this} \setminus \text{next is a cx mfld of codim } (r-k+1) \\ \quad \cup \\ \text{rank} \le k-2 \\ \quad \cup \\ \quad \vdots \end{array}$$



then f^{-1}(each stratum ∖ next) is a submfld of X
w/ correct codim.

notice that Z is not necessarily a manifold but it has an open dense set that is a manifold,

so $\displaystyle\int_X \alpha \wedge c_{r-k+1}(E) = \int_Z \alpha$ is legit.

$\deg \alpha = \dim X - 2(r-k+1) = \dim Z.$

---

could emulate what we did before:

on $X \setminus Z$, have k indep sections of E

$\rightsquigarrow E \cong \mathcal{O}^k \oplus E'$

choose a compat connection on $X \setminus Z$

and any connection on a nbd of Z

... want to know something like

$C_{top}(N_{Z/X}) = c_{r-k+1}(E)|_Z$

but worry about Z having singularities...

could make it work, but let's do something else.

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for definiteness, let $r=3$ and $k=2$

so two sections $s, t \in \Gamma(E)$

lin. dependent on $Z \subset X$ of real codim 4.

claim that Z is PD to $c_2(E)$.

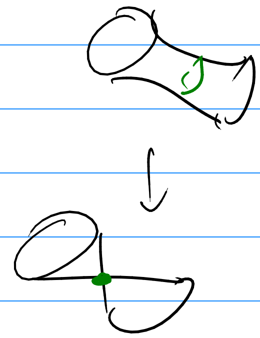
in $\mathbb{P}E$, which is a $\mathbb{CP}^2$-bundle whose fiber over $x \in X$ is $\mathbb{P}(E_x)$

$\pi \downarrow$

X

$= \{1\text{-dim'l subspaces of } E_x\}$

look at $\tilde{Z} = \{ x \in X,\ \ell' \subset E_x \mid s(x), t(x) \in \ell \subset E_x \}$.



$$\begin{array}{ccc} \tilde{Z} & \subset & \mathbb{P}E \\ \downarrow & & \downarrow \pi \\ Z & \subset & X \end{array}$$

iso where $s(x)$ and $t(x)$ span a 1-dim'l subspace of E_x but not 0-dim'l.

Worksheet: $\tilde{Z}$ is a manifold.

for α a closed form of deg = dim Z on X,

$$\int_Z \alpha = \int_{Z'} \pi^* \alpha$$

want ||

$$\int_X \alpha \wedge c_2(E)$$

$$\int_{\mathbb{P}E} \pi^*\alpha \wedge c_4(Q^2)$$

$$= \int_X \alpha \wedge \pi_* c_4(Q^2)$$

want: $\pi_* c_4(Q^2) = c_2(E)$.

$\tilde{Z} \subset \mathbb{P}E$ is cut out by a $\pitchfork$ section of Q^2, where Q is defined by the exact seq.

$$0 \to \mathcal{O}_\pi(-1) \to \pi^* E \to Q \to 0$$

$\hookrightarrow$ at $x \in X$, $\ell \in \mathbb{P}E_x$, fiber of $\mathcal{O}_\pi(-1)$ is ℓ,

fiber of $\pi^* E$ is E_x,
and fiber of Q is E_x/ℓ.

$s, t \in \Gamma(E)$ on X

determine sections of $\pi^* E$ on $\mathbb{P}E$

$\rightsquigarrow$ sections of Q on $\mathbb{P}E$

$\hookrightarrow$ those vanish at $x \in X$, $\ell \in \mathbb{P}E_x$
iff $s(x)$ and $t(x)$ are 0 in E_x/ℓ,
iff $s(x), t(x) \in \ell$.

s, t are gen in the sense we asked for
$\rightsquigarrow$ then two sections of Q
give a transverse section of $Q \oplus Q$
that cuts out $\tilde{Z}$

so $\tilde{Z}$ is PD to $c_4(Q \oplus Q)$.

(clear it up next time)