Last time:

$E \to X$ cx v.b. of rank r

$s_1, \dots, s_k \in \Gamma(E)$ "generic"

$Z \subset X$ the set where they're lin. dep.

$$\operatorname{codim}_{\mathbb{C}} Z = (k-(k-1))(r-(k-1)) = 1 \cdot (r-k+1)$$

$$\operatorname{codim}_{\mathbb{R}} Z = 2(r-k+1)$$

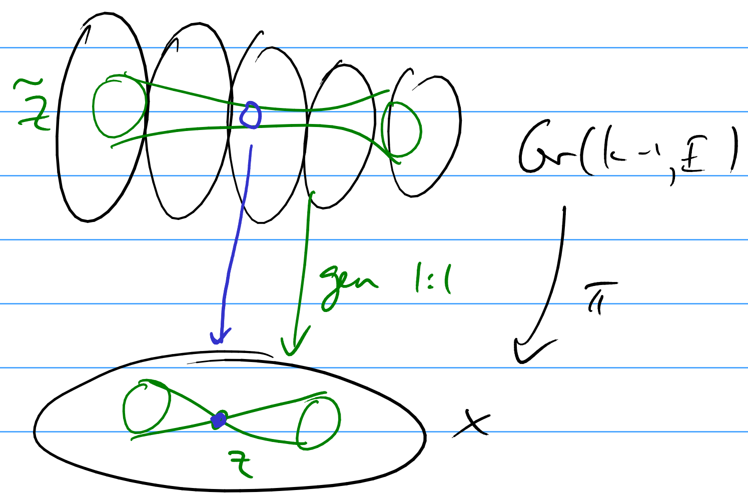
claim: Z is PD to $c_{r-k+1}(E)$.

in general, Z is singular along the set where $s_1, \dots, s_k$ span a space of dim $\le k-2$

consider $\tilde{Z} = \left\{ x \in X, \ W^{k-1} \subset E_x \text{ containing } \operatorname{span}(s_1(x) \dots s_k(x)) \right\} \subset Gr(k-1, E)$

$\uparrow$ bundle over X whose fiber over x is $Gr(k-1, E_x)$

challenge problem: $\tilde{Z}$ is smooth.



on Gr, tautological sub-bundle $\underset{\substack{\uparrow \\ \text{rank } k-1}}{S} \subset \pi^* E$

tautological quot. bundle $Q = \pi^* E / S$ rank $= r-k+1$

fiber of Q at $x \in X$, $W^{k-1} \subset E_x$
is E_x / W.

$s_1, \ldots, s_k$ give sections of $\pi^* E$,
hence sections of Q — call them $\tilde{s}_1, \ldots, \tilde{s}_k \in \Gamma(Q)$

at $x \in X$, $W \subset E_x$ in $Gr(k-1, E)$
$\tilde{s}_i$ vanishes iff $s_i(x) \in W \subset E_x$.

so $\tilde{Z}$ is cut out of $Gr((k-1, E)$
by one section of $Q^{\oplus k}$
check: codim $\tilde{Z}$ in Gr $=$ rank $Q^{\oplus k}$
in fact the section of $Q^{\oplus k}$ is transverse
iff $s_1, \ldots, s_k \in \Gamma(E)$ were generic...

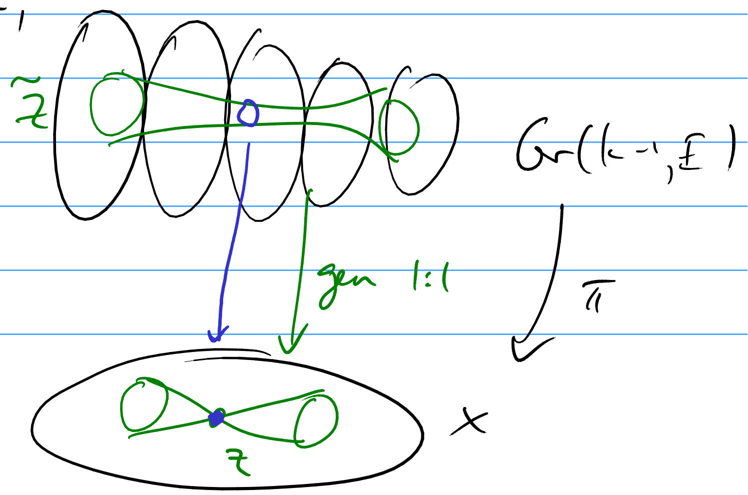
---

for a closed form α on X
with deg $\alpha =$ dim Z,

$$\int_Z \alpha = \int_{Z'} \pi^* \alpha$$

$$= \int_{Gr} \pi^* \alpha \wedge c_{top}(Q^{\oplus k})$$

$$= \int_X \alpha \wedge \pi_* c_{top}(Q^{\oplus k})$$



claim reduces to $\pi_* c_{top}(Q^{\oplus k}) = c_{r-k+1}(E)$

do it for $r=3$ $k=2$

2 sections of a rank-3 bundle E

Z = where they're lin. dep. $\operatorname{codim}_{\mathbb{C}} Z = 2$

$\operatorname{codim}_{\mathbb{R}} Z = 4$

$Gr(1,E)$ is a $\mathbb{CP}^2$-bundle over X

rank $Q = 2$

$c_{top}(Q^{\oplus 2}) = c_4(Q^{\oplus 2}) = c_2(Q)^2$

$\hookrightarrow c(Q^{\oplus 2}) = c(Q)^2 = (1 + c_1 + c_2)^2$

$= 1 + \cdots + c_2^2$

exact seq. on $Gr(1,E) = \mathbb{P}E$

$0 \to S \to \pi^* E \to Q \to 0$

$\leftrightarrows$ line bundle, aka $\mathcal{O}(-1)$

So smoothly, $\pi^* E = S \oplus Q$

$\pi^* c_3(E) = \underline{c_1(S)\, c_2(Q)}$

$\left(\pi^* c_2(E) = c_1(S)\, c_1(Q) + c_2(Q) \right) \cdot c_2(Q)$

$\pi^* c_2(E) \cdot c_2(Q) = \underline{c_1(S)}\, c_1(Q)\, \underline{c_2(Q)} + c_2(Q)^2$

$= \pi^* c_3(E)\, c_1(Q) + c_2(Q)^2$

apply π_*:

$$c_2(E) \cdot \underbrace{\pi_* c_2(Q)}_{} = c_3(E)\, \cancel{\pi_* c_1(Q)}^{0} + \pi_*\left(c_2(Q)^2\right)$$

if this $= 1$ then we win.

on one $\mathbb{P}^2$ fiber, $\int c_2(Q) = 1$

$$0 \to S \to \mathcal{O}_{\mathbb{P}^2}^3 \to Q \to 0$$

if $h = -c_1(S)$ then $c(Q) \cdot (1-h) = c(\mathcal{O})^3 = 1$

so $c(Q) = 1 + h + h^2$

$$\int h^2 = 1$$

(or: gen section of Q cuts out 1 point in $\mathbb{CP}^2$.)