Chern-Weil story:

\[ G = \text{Lie group} \]
\[ \mathfrak{g} = \text{Lie alg.} \quad (\text{in dim } n \text{ use } \mathbb{R} \text{ or } \mathbb{C}) \]
\[ \mathfrak{g}^* = \text{polynomial on } \mathfrak{g}. \quad \text{aka } \mathbb{R}[\mathfrak{g}] \text{ or } \mathbb{C}[\mathfrak{g}] \]

Then \( G \subset \mathfrak{g}^* \) "by conjugation":

\[ g \cdot f \alpha = f(\exp g^{-1}) \]

\[ \mathfrak{g}^* \] is invariant subspace.

Examples: if \( G = GL_n \mathbb{R} \) then

\[ \mathfrak{g}^* = \text{polynomial ring on } e_1, \ldots, e_n \]

where

\[ e_i = \text{ith elem. symm. function of eigenvals.} \]

\[ = \text{ith coeff of char. polynomial}. \]

\[ A \in g \implies \det(eI - A) = \text{same with } GL_n \mathbb{C} \]

\[ = t^n - \sum A t^{n-2} + \cdots \]

\[ = \text{same with } O(n) \]

but \( SO(n) \) also has the Pfaffian.

For a principal \( G \)-bundle \( P \to X \), manifold

Chern-Weil theory constructs a hom.

\[ \mathfrak{g}^* \to \text{H even }\mathbb{R}(X, \mathbb{R} \text{ or } \mathbb{C}) \]

\[ f \mapsto f(\text{curvature}) \]
Fix $G$. A degree-$d$ poly $f \in \mathbb{R}^d$ gives a natural transformation of pointed principal $G$-bundles on manifolds $\rightarrow$ sets.

First functor is represented by a space $BG$.

Principal $G$-bundles on $X = \text{Hom}(X, BG)$.

Awkward: $BG$ is not a (finite-dim) manifold.

Yoneda's lemma: Set of natural transformations $\text{Yon}(\cdot, BG) \rightarrow \text{Yon}(\cdot, BG) = H_{dR}(BG)$.

More precise: For a manifold $X$, the map $\{\text{principal } G\text{-bundles on } X\} \rightarrow H^{2d}_{dR}(X)$.

$\psi : EG \rightarrow \text{Yon}(\cdot, BG)$ is $\psi$ of $i$'s class in $H_{dR}(BG)$.

$\text{Hom}(X, BG) \rightarrow \text{Yon}(\cdot, BG)$.

So we suspect that $H^*_{dR}(BG) = \mathbb{R}^d$.

Thus it's true.
Point of our Yoneda argument:

given real or complex vector bundle

oriented \quad \text{unoriented}

we know how to construct lots of char classes.

How do we know when we've found them all?

a char. class is a nat. transf. from
\[
\{\text{vector bundles on } X\} \longrightarrow \text{some } H^i(X)
\]

by Yoneda, the set of all such nat. transf. is
\[
H^i(BGL_n R) \quad \text{or} \quad H^i(BGL_n^+ C)
\]
\[
\quad \text{or} \quad H^i(BGL^+_n R)
\]

\[
BGL_n R = BGL(n) = Gr(n, \mathbb{R}^n)
\]
\[
= \lim Gr(n, n) \hookrightarrow Gr(n, n+1)
\]
\[
\hookrightarrow Gr(n, n+2)
\]
\[
\quad \text{vector universal bundle is the tautological bundle } S_n
\]
\[
\text{universal principal bundle } EGL_n \text{ is the frame bundle of that,}
\]
\[
BGL_n
\]
\[\{v_1, \ldots, v_n \in \mathbb{R}^n \mid \text{lin. indep.}\}\]
\[ BGL_n(C) = BGL(R) = \text{Gr}(n, C^n) \]

\[ BGL_n^+(R) = BSO(n) = \text{Gr}_{\text{ori.+}}(n, R^n) \xrightarrow{\text{2.5}} \text{Gr}(n, R^n) \]

Answer: \[ H^\ast(BGL_n(C), \mathbb{Z}) = \mathbb{Z}[c_1, c_2, \ldots, c_n] \]

where \( c_i \in H^{2i} \) Chern classes

\[ H^\ast(BGL_n(R), \mathbb{Z}[\frac{1}{2}]) = \mathbb{Z}[\frac{1}{2}][p_1, p_2, \ldots, p_{2n}] \]

where \( p_i \in H^{4i} \) Pontryagin classes

if \( n \) is odd, \( H^\ast(BGL_n^+(R), \mathbb{Z}[\frac{1}{2}]) \) is the same

if \( n \) is even, also have the Euler class \( X \in H^\ast(BGL_n^+(R), \mathbb{Z}[\frac{1}{2}]) \)

\[ X^2 = p_{2n} \]

can also do \( H^\ast(BGL_n(R), \mathbb{Z}) \) or Stiefel-Whitney classes \( H^\ast(BGL_n(R), \mathbb{Z}) \) or Pontryagin classes.

Read Milnor+Stasheff

Beckstein (S-W classes)

or

Lawson+Michelson

"Spin Geometry"

Appendix B.

\[ 0 \rightarrow \mathbb{Z}^2 \rightarrow \mathbb{Z}_2 \rightarrow \mathbb{Z}_2 ightarrow 0 \]

\[ \beta: H^\ast(K, \mathbb{Z}_2) \rightarrow H^{\ast+1}(K, \mathbb{Z}) \]

Warning: Ring structure is complicated.