Chern-Weil story:

G = Lie group
$\mathfrak{g}$ = Lie alg. (fin dim'l v.s. over $\mathbb{R}$ or $\mathbb{C}$)
R = polynomials on $\mathfrak{g}$.
aka $\mathbb{R}[\mathfrak{g}]$ or $\mathbb{C}[\mathfrak{g}]$

then $G \circlearrowright R$ "by conjugation":

$$g \cdot f(x) = f(gxg^{-1})$$

R^G = invariant subspace.

Examples: if $G = Gl_n\mathbb{R}$ then

R^G = polynomial ring on $e_1, \dots, e_n$

where e_i = i^{th} elem. symm. function of eigenvals.
= i^{th} coeff of char. polynomial.

$A \in \mathfrak{g} \mapsto \det(tI - A) = t^n - \underline{\operatorname{tr} A}\, t^{n} + \dots \pm \underline{\det A}$

same with $Gl_n\mathbb{C}$

same with $O(n)$

but $SO(n)$ also has the Pfaffian.

For a principal G-bundle $P \to X$, ← manifold
Chern-Weil theory constructs a hom.

$$R^G \longrightarrow H^{even}_{dR}(X, \mathbb{R} \text{ or } \mathbb{C})$$
$$f \longmapsto f(\text{curvature})$$

fix G.

a deg-d poly $f \in \mathbb{R}^G$ gives a nat. transformation

$$\begin{tikzcd}
\text{principal } G\text{-bundles} : \text{manifolds}^{op} \arrow[r] \arrow[d, "\text{nat. transf.}"] & \text{pointed sets} \\
H^{2d}_{dR} : \text{manifolds}^{op} \arrow[r] & \text{vector spaces}
\end{tikzcd}$$

first functor is represented by a space BG

principal G-bundles on X $=$ $\mathrm{Hom}(X, BG)$

awkward: BG is not a (finite-dim'l) manifold

Yoneda's lemma: Set of natural transformations
$= H^{2d}_{dR}(BG)$

more precise: for a mfld X, the map

$$\begin{tikzcd}
\{\text{principal } G\text{-bundles on } X\} \arrow[r, "\cong"] & H^{2d}_{dR}(X) \\
\mathrm{Hom}(X, BG) \arrow[u] &
\end{tikzcd}$$

$\varphi^*(EG)$

$\varphi : X \to BG \mapsto \varphi^*$ of my class in $H^{2d}_{dR}(BG)$

So we suspect that $H^*_{dR}(BG) \cong \mathbb{R}^G$.

Thm: it's true.

Point of our Yoneda argument:

given real or complex vector bundle,

oriented unoriented

we know how to construct lots of char classes.

How do we know when we've found them all?

a char. class is a nat transf. from

$\{\text{vector bundles on } X\} \longrightarrow \text{some } H^*(X)$

by Yoneda, the set of all such nat. transformations

is $H^*(BGL_n\mathbb{R})$ or $H^*(BGL_n\mathbb{C})$

or $H^*(BGL_n^+\mathbb{R})$ ...

$BGL_n\mathbb{R} = BO(n) = Gr(n, \mathbb{R}^\infty)$

$= \lim Gr(n,n) \hookrightarrow Gr(n,n+1)$

$\hookrightarrow Gr(n,n+2)$

$\hookrightarrow \cdots$

universal vector bundle is the tautological bundle S.

univ. principal bundle $\begin{matrix} EGL_n \\ \downarrow \\ BGL_n \end{matrix}$ is the frame bundle of that,

aka the Stiefel manifold

$\{e_1, \ldots, e_n \in \mathbb{R}^\infty \mid \text{lin. indep.}\}$

$BGL_n\mathbb{C} = BU(n) = Gr(n, \mathbb{C}^\infty)$

$BGL_n^+\mathbb{R} = BSO(n) = Gr_{orient}(n, \mathbb{R}^\infty) \xrightarrow{2:1} Gr(n, \mathbb{R}^\infty)$

Answer: $H^*(BGL_n\mathbb{C}, \mathbb{Z}) = \mathbb{Z}[c_1, c_2, \ldots, c_n]$

where $c_i \in H^{2i}$ Chern classes

$H^*(BGL_n\mathbb{R}, \mathbb{Z}[1/2]) = \mathbb{Z}[1/2][p_1, p_2, \ldots, p_{\lfloor n/4 \rfloor}]$

where $p_i \in H^{4i}$ Pontryagin classes

if n is odd, $H^*(BGL_n^+\mathbb{R}, \mathbb{Z}[1/2])$ is the same

if n is even, also have the Euler class

$\chi \in H^n(BGL_n^+\mathbb{R}, \mathbb{Z}[1/2])$

$\chi^2 = p_{n/2}$

can also do $H^*(BGL_n\mathbb{R}, \mathbb{Z}_2) \rightsquigarrow$ Stiefel-Whitney classes

$H^*(BGL_n\mathbb{R}, \mathbb{Z}) \rightsquigarrow$ Pontryagin classes,

Bockstein (S-W classes)

$0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}_2 \to 0$

$\beta: H^i(X, \mathbb{Z}_2) \to H^{i+1}(X, \mathbb{Z})$

warning: Ring structure is complicated.

Read Milnor + Stasheff

OR Lawson + Michelsohn "Spin Geometry" Appendix B.