Čech coho. with coeffs in non-Abelian G

Worksheet: if G is an Ab. group,
$\mathcal{U}$ = open cover of X,

$$\check{C}^n(\mathcal{U}, G) = \text{tuples } \xi_{i_0 \cdots i_n} : U_{i_0} \cap \cdots \cap U_{i_n} \to G$$

with some $d : \check{C}^n \to \check{C}^{n+1}$ $\quad d^2 = 0$

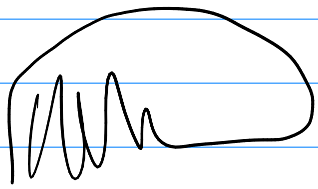
take $\check{H}^n(\mathcal{U}, G) = \ker d / \operatorname{im} d$

$\check{H}^n(X, G) = \varinjlim$ over all open covers.

the cover $\mathcal{U}$ is good if all intersections
$U_{i_0} \cap \cdots \cap U_{i_m}$ are empty or contractible

↳ used in the proof that
$\check{H}^n(X, \text{locally const maps to } G) = H^n_{\text{sing}}(X, G)$
when X is a CW complex

↳ or that $\check{H}^n(X, \text{locally const maps to } \mathbb{R})$
$= H^n_{dR}(X)$ when X is a mfld.
(read Bott + Tu)

but if $X =$  has $\check{H}^1(X, \mathbb{Z}) = \mathbb{Z}$

whereas $H^1_{\text{sing}}(X, \mathbb{Z}) = 0$

Maybe you had time to play with

$\check{H}^i(\text{manifold, smooth maps to } \mathbb{R}) = 0$ for $i > 0$

professional way: choose a partition of 1 $\rho_i : U_i \to [0,1]$

define $P : \check{C}^n(\mathcal{U}, \mathbb{R}) \longrightarrow \check{C}^{n-1}(\mathcal{U}, \mathbb{R})$

$$\xi \longmapsto P\xi$$

$$(P\xi)_{i_1 i_2 \dots i_n} := \sum_{i_0} \rho_{i_0} \xi_{i_0 i_1 \dots i_n}$$

check that $Pd + dP = 1$.

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if G is not abelian then

$\check{H}^0(X, G) = \text{maps } X \to G$

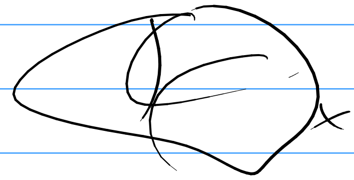
it's a group.

$\check{H}^1(X, G) =$ principal G-bundles.

use $\psi_{ij} : U_i \cap U_j \to G$

as transition functions

$\psi_{ij} \cdot \psi_{ik}^{-1} \cdot \psi_{jk} = 1$ so it works.



if ψ_{ij}' came from another trivialization

then we'd have $\psi_{ij}' = \varphi_j \psi_{ij} \varphi_i^{-1}$

so same in $\check{H}^1$.

$\check{H}^1(X, G) =$ pointed set.

$\check{H}^2(X, G) =$ not defined if G not Ab.

reminiscent: if $X \supset A$, rel. homotopy groups

$\pi_i(X,A)$ is ab. for $i \geq 3$

$\pi_2(X,A)$ is a group
$\pi_1(X,A)$ is a pointed set
$\pi_0(X,A)$ is undefined.

also related: if G is Abelian then
BG is close enough to being a group
that we can define BBG, $BBBG$, ...

but not if G is non-Abelian.

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let X be a manifold, take $\check{H}$ with smooth maps to G.
↳ if we did locally const maps ...

$H^1(X, O(n))$ = real vector bundles of rank n

$H^1(X, SO(n))$ = oriented real v.b's.

$H^1(X, U(n))$ = complex vector bundles of rank n

get v.b's with flat connection!

$H^1(X, \mathbb{Z}_2)$ = double covers of X.

Long exact sequence? if G is a Lie group and $H \subset G$ is a closed subgroup then we get

$$0 \to H^0(X,H) \to H^0(X,G) \to H^0(X, G/H) \to H^1(X,H) \to H^1(X,G)$$

$\hookrightarrow$ not a group, only a space

connecting hom is great:
the map $G \to G/H$ is a principal H-bundle

given a map $X \to G/H$, pull it back to get a principal H-bundle on X

Example: $H = O(n)$ $\quad G = O(n+1)$ $\quad G \circlearrowright S^n \subset \mathbb{R}^{n+1}$

$$A \longmapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$$

stab of $\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$ is H

so $G/H \cong S^n$

$$\begin{tikzcd} O(n) \arrow[r, hook] & O(n+1) \arrow[d] \\ & S^n \end{tikzcd}$$

(this is the frame bundle of TS^n)

$$0 \to \text{maps}(X \to O(n)) \to \text{maps}(X \to O(n+1)) \to \text{maps}(X \to S^n)$$

$f \longmapsto$

$\to$ rank n vector bundles $\to$ rank $n+1$ vector bundles.

$\longmapsto f^* TS^n$ $\qquad E \longmapsto E \oplus \mathcal{O}$

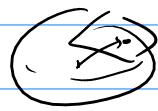
if $E \oplus \mathcal{O}_X \xrightarrow[\varphi]{\sim} \mathcal{O}_X^{n+1}$

then $\exists$ map $f: X \to S^n$ s.t. $E \cong f^* TS^n$

$\forall\ x \in X$, φ gives an iso $E_x \oplus \mathbb{R} \xrightarrow{\varphi_x} \mathbb{R}^{n+1}$

define $f(x) = \varphi_x(0,1) \big/ \text{length} \in S^n \subset \mathbb{R}^{n+1}$

then $T_{f(x)} S^n = f(x)^{\perp} \subset \mathbb{R}^{n+1} \cong E_x$



if $H \subset G$ is normal then G/H is a group, and

$$0 \to H^0(X,H) \to H^0(X,G) \to H^0(X, G/H)$$

$$\to H^1(X,H) \to H^1(X,G) \to H^1(X, G/H)$$ is exact.

example: $H = SO(n)$ $\quad G = O(n)$ $\quad G/H = \mathbb{Z}/2$

$$1 \to SO(n) \hookrightarrow O(n) \xrightarrow{\det} \{\pm 1\} \to 1$$

$$0 \to \text{maps}(X \to SO(n)) \to \text{maps}(X \to O(n)) \to \text{maps}(X \to \mathbb{Z}/2)$$

$\to$ oriented v.b.'s $\longrightarrow$ vector bundles $\xrightarrow{w_1}$ double covers of $H^1(X, \mathbb{Z}_2)$

first Stiefel-Whitney class

if H is in the center of G, then

$$0 \to H^0(X,H) \to H^0(X,G) \to H^0(X, G/H)$$

$$\to H^1(X,H) \to H^1(X,G) \to H^1(X, G/H)$$

$$\to H^2(X,H)$$ is exact.

example: know that $\pi_1(SO(n)) = \mathbb{Z}_2$ for $n \geq 3$
double cover is $Spin(n)$

$$1 \to \mathbb{Z}/2 \to Spin(n) \to SO(n) \to 1$$

maps $(X \to SO(n))$

$\to H^1(X, \mathbb{Z}/2) \to$ bundles w/ Spin str $\to$ oriented bundles $\xrightarrow{w_2}$

$\to H^2(X, \mathbb{Z}/2)$

first Chern class:

$$1 \to SU(n) \to U(n) \xrightarrow{\det} U(1) = S^1 \to 0$$

$c_1(E)$
$= c_1(\det E)$

$$H^1(X, SU(n)) \to \underset{\text{cx v.b.'s}}{H^1(X, U(n))} \to H^1(X, U(1))$$

we also have $1 \to \mathbb{Z} \to \mathbb{R} \to S^1 \to 0$
$x \longmapsto e^{2\pi i x}$

$$\overset{0}{\cancel{H^1(X,\mathbb{R})}} \to \underset{\text{cx line bundles}}{H^1(X, U(1))} \xrightarrow{c_1} H^2(X, \mathbb{Z}) \to \overset{0}{\cancel{H^2(X,\mathbb{R})}}$$