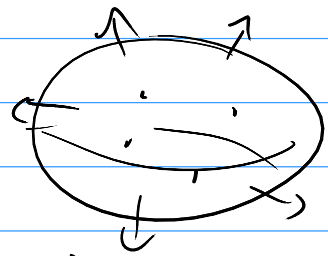
Electromagnetism + Differential Forms

two vector fields on $\mathbb{R}^3$:

electric field $E = (E_1, E_2, E_3)$
magnetic field $B = (B_1, B_2, B_3)$

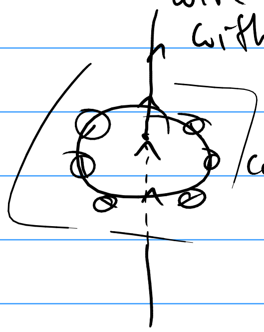
Gauss's law: $\iint_{\text{closed surface}} E \cdot d\vec{n}$ = charge inside



$\iint_{\text{closed surface}} B \cdot d\vec{n} = 0$

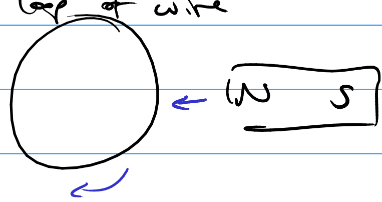
(no magnetic charges)

Ampère's law: $\int_{\text{closed loop}} B$ = current through + flux of $\frac{\partial E}{\partial t}$



Faraday's law of induction:

$\int_{\text{closed loop}} E$ = flux of $-\partial B/\partial t$ through surface whose boundary is the loop



Suppressing constants
$\varepsilon_0 \quad \mu_0 \quad c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$

choose units to make $= 1$

Differential formulation:

Gauss $\rightsquigarrow$ $\text{div}\, E = \text{charge density}$

$$\text{div}\, E = \nabla \cdot E = \frac{\partial E_0}{\partial x} + \frac{\partial E_1}{\partial y} + \frac{\partial E_2}{\partial z}$$

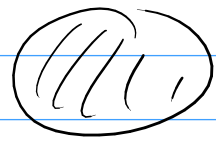
$$\left(\iiint_{\text{solid}} \text{div}\, E = \iint_{\text{boundary}} E \cdot \vec{n} \right.$$

$$\left. \text{and} \quad \iiint \text{charge density} = \text{total charge} \right)$$

$$\text{div}\, B = 0$$

Ampère's law $\rightsquigarrow$ $\text{curl}\, B = \text{current density} + \partial E / \partial t$

$$\left(\iint_{\text{surface}} \text{curl}\, B = \int_{\text{boundary}} B \right)$$



Faraday's law $\rightsquigarrow$ $\text{curl}\, E = -\partial B / \partial t$

(if B is const then $\nabla \times E = 0$

$\rightsquigarrow E = \nabla \phi$ $\quad \phi$ a function, "electrical potential")

Maxwell's equations

translate to differential forms in 3D:

vector field $E = (E_1, E_2, E_3)$

$\hookrightarrow$ 1-form $E = E_1\,dx + E_2\,dy + E_3\,dz$

1-form $= g(\text{v.f.}, -)$ $\quad g =$ std Riem. metric.

$B \rightsquigarrow$ 2-form $B = B_1\,dy\wedge dz + B_2\,dz\wedge dx + B_3\,dx\wedge dy$

2-form $=$ v.f. $\lrcorner\,(dx\wedge dy\wedge dz)$

Gauss's law for magnetism: $\int_{\text{closed surface}} B = 0$

or $dB = 0$.

(div. of a v.f. $\cdot\, dx\wedge dy\wedge dz = d$ (corresp 2-form))

Faraday: $dE = -\partial B/\partial t$

(curl of a v.f. $\rightsquigarrow$ 2-form
$= d$ (corresp 1-form))

Hodge star: $\ast\, dx = dy\wedge dz \quad \ast\, dy = dz\wedge dx$

$\ast\, dz = dx\wedge dy$

in general $\alpha\wedge\ast\,\alpha = |\alpha|^2\cdot dx\wedge dy\wedge dz$

$*E$ is a 2-form $E_1\, dy \wedge dz + E_2\, dz \wedge dx + E_3\, dx \wedge dy$
$*B$ is a 1-form $B_1\, dx + B_2\, dy + B_3\, dz$

Gauss's law: $d(*E) = \underline{\text{charge density}}$
↳ naturally a 3-form
bec. you $\iiint$ to get charge.

Ampère's law: $d(*B) = \underline{\text{current density}} + \partial(*E)/\partial t$
↳ 2-form bec. you $\iint$ to find out how much current is passing thru surface.

Differential forms in 4D:

replace E and B with

$$F = E \wedge dt + B \quad \text{on } \mathbb{R}^4$$

$$= E_1\, dx \wedge dt + E_2\, dy \wedge dt + E_3\, dz \wedge dt$$
$$+ B_1\, dy \wedge dz + B_2\, dz \wedge dx + B_3\, dx \wedge dy$$

Gauss for magnetism + Faraday become
$dF = 0$

$$dF = dE \wedge dt + (3D\ dB) + \frac{\partial B}{\partial t} \wedge dt$$

Gauss's law for electricity + Ampère become

$$d(*F) = (\text{charge density})\, dx \wedge dy \wedge dz$$
$$+ \underbrace{(\text{current density})}_{\hookrightarrow J_1\, dy \wedge dz \wedge dt \ + \ 2 \text{ more}} \wedge dt$$

$*$ is taken w.r.t. Lorentz metric

$$g = dx^2 + dy^2 + dz^2 - dt^2$$

pseudo-Riemannian, sig. $+++-$

$$* \, dx \wedge dy = \pm \, dz \wedge dt \quad \text{etc.}$$

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in absence of charges and currents,

$$dF = 0 \quad \text{and} \quad d(*F) = 0$$

"harmonic 2-form"

if we're on a closed oriented Riemannian mfld
then every class in H^*_{dR}
has a unique harmonic representative.
$\hookrightarrow$ Hodge theory

4-potential:

$dF = 0$, so we can write

$F = dA$ for some 1-form

$$A = \underbrace{A_1\,dx + A_2\,dy + A_3\,dz}_{\text{magnetic vector potential}} + \phi\,dt$$

ϕ ↳ electrical potential, at least if $\partial B / \partial t = 0$

change $A \rightsquigarrow A + d\psi$ doesn't affect F, "gauge transformation"

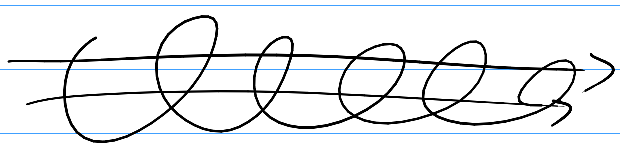
often said that we should interpret A as a connection on a principal $U(1)$-bundle

on $\mathbb{R}^4$, every $U(1)$-bundle is trivial, so why do this?

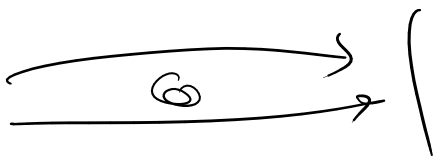
① holonomy of this connection

$$\int_{\text{closed loop}} e^{iA}$$ is physically meaningful.

Aharonov-Bohm effect:

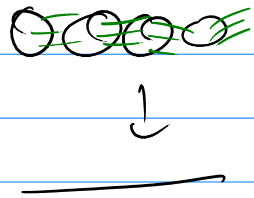


solenoid



(2) Kaluza-Klein story:

take $$\begin{array}{c} P = \mathbb{R}^4 \times U(1) \\ \downarrow \\ \mathbb{R}^4 \end{array}$$



Lorenz metric downstairs + connection on P

$\rightsquigarrow$ metric upstairs

write eqns of gen. relativity upstairs

$\longleftrightarrow$ eqns of gen relativity + Maxwell's eqns downstairs.

(3) when you go to do weak + strong force then it becomes more clear that you want connections on principal $SU(2)$ and $SU(3)$ - bundles.

Something to read: Bott "On some recent connections between math + physics." (1985)