Worksheet 1

Math 607, Connections and Characteristic Classes

Monday, January 4, 2021

Don't turn this in, just work through it together, and it's ok if you don't finish. Today we'll review vector bundles and fix some notation. If you want to read the basics, I recommend Chapter 10 of Lee's *Introduction to smooth manifolds (2nd edition)*.

0. Introduce yourself to your colleague(s). Did they have a good break?

1. Let X be a smooth manifold. Recite the definition of a *smooth vector bundle* $p\colon E \to X$ of rank r, or ask your colleague(s) to explain it to you.

2. The *fiber* of E over a point $x \in X$ is $E_x := p^{-1}(x)$. It is an r-dimensional vector space.

   A *(global) section* of E is a smooth map $s\colon X \to E$ such that $p \circ s = 1$. Thus for every $x \in X$ we have $s(x) \in E_x$.

   Our favorite vector bundles are the tangent bundle T_X, the Möbius bundle over S^1, and the trivial bundle $X \times \mathbb{R}$, which as an algebraic geometer I can't help calling $\mathcal{O}_X$. (Soon we'll talk about the tautological sub- and quotient bundles on a Grassmannian.) Convince yourselves that a section of T_X is the same as a vector field, and a section of $\mathcal{O}_X$ is the same as a smooth function $X \to \mathbb{R}$. Draw the Möbius bundle and a section of it.

3. Maps of vector bundles. If $p\colon E \to X$ and $q\colon F \to X$ are two vector bundles, we consider smooth maps $f\colon E \to F$ such that $q \circ f = p$, so for all $x \in X$ we get a map $E_x \to F_x$, and we require this to be linear. Convince yourselves that a map $E \to F$ is the same as a section of $E^* \otimes F$.

4. Tensoriality. Let $C^\infty(X)$ be the ring of smooth functions on X, and for a vector bundle E, let $\Gamma(E)$ be the set of global sections, which is a $C^\infty(X)$-module. A map of vector bundles $E \to F$ determines a $C^\infty(X)$-linear map $\Gamma(E) \to \Gamma(F)$. Show that the converse is true as well: if a map $\Gamma(E) \to \Gamma(F)$ is $C^\infty(X)$-linear, then it comes from a map of vector bundles $E \to F$.