Worksheet 10

Math 607, Connections and Characteristic Classes

Friday, February 26, 2021

This is about Pfaffians of skew-symmetric matrices.

1. Let $V = \mathbb{R}^{2n}$. A 2-form $\omega \in \Lambda^2 V^*$ determines a map

$$A: V \to V^*$$

$$v \mapsto \omega(v, -)$$

which is skew-symmetric, meaning that $A^T: V^{**} \to V^*$ equals $-A$.
We have the determinant

$$\Lambda^{2n} A: \Lambda^{2n} V \to \Lambda^{2n} V^*,$$

or for compactness we can also write

$$\det A: \det V \to \det V^*.$$

We will define

$$\text{Pf}(A) = \frac{1}{n!} \omega^\wedge n \in \Lambda^{2n} V^* = \det V^*.$$

The reason for the $n!$ will become clear later.

Because $\det V$ and $\det V^*$ are 1-dimensional, we can regard $\det A$ as an element of $(\det V^*)^\otimes 2$, so it’s reasonable to hope that $\det A = \text{Pf}(A)^2$.

Discuss any questions that you have about this set-up.

2. Let $W$ be another copy of $\mathbb{R}^{2n}$, and let $B: W \to V$. Then the 2-form $B^* \omega \in \Lambda^2 W^*$ corresponds to the map

$$B^T \circ A \circ B: W \to V \to V^* \to W^*.$$

As matrices, we know that $\det(B^T AB) = (\det B)^2 \cdot \det A$. Convince yourselves that this is ok in our invariant set-up, where

$$\det B, \det B^T \in \det W^* \otimes \det V$$

$$\det A \in (\det V^*)^\otimes 2$$

$$\det(B^T AB) \in (\det W^*)^\otimes 2.$$
Then convince yourselves that
\[
Pf(B^T \circ A \circ B) = \det B \cdot Pf(A),
\]
because
\[
(B^* \omega)^\wedge n = B^*(\omega^\wedge n)
\]
and \(B^*: \Lambda^{2n} V^* \to \Lambda^{2n} W^*\) is \(\det B\).

3. In a skew-symmetric analogue of Sylvester’s law of inertia, we can choose a basis
\[
e_1, e_2, \ldots, e_{2n}
\]
in which \(\omega\) becomes
\[
e_1^* \wedge e_2^* + \cdots + e_{2m-1}^* \wedge e_{2m}^*
\]
for some \(m \leq n\), and \(A\) becomes
\[
\begin{pmatrix}
0 & 1 & & \\
-1 & 0 & & \\
& & \ddots & \\
& & 0 & 1 & \\
& & -1 & 0 & \\
& & & & \ddots & \\
& & & & & 0
\end{pmatrix}
\]
where the number of \(2 \times 2\) blocks is \(m\) and the number of zeros is \(2n - 2m\).
Think about this.

4. View this choice of basis as an isomorphism \(B: \mathbb{R}^{2n} \to V\).
   If \(m < n\) then \((B^* \omega)^\wedge n = 0\) and \(\det(B^T AB) = 0\), so \(\det A = 0 = (\Pf A)^2\).
   If \(m = n\) then \((B^* \omega)^\wedge n\) is \(n!\) times the standard volume form \(e_1^* \wedge \cdots \wedge e_{2n}^*\),
   and \(\det(B^T AB) = 1\), so by \#2 we get \(\det A = (\det B)^{-2} = (\Pf A)^2\).