Worksheet 10

Math 607, Connections and Characteristic Classes

Friday, February 26, 2021

This is about Pfaffians of skew-symmetric matrices.

1. Let $V = \mathbb{R}^{2n}$. A 2-form $\omega \in \Lambda^2 V^*$ determines a map

$$\begin{aligned} A\colon V &\to V^* \\ v &\mapsto \omega(v, -) \end{aligned}$$

which is skew-symmetric, meaning that $A^\top : V^{**} \to V^*$ equals $-A$.

We have the determinant

$$\Lambda^{2n} A\colon \Lambda^{2n} V \to \Lambda^{2n} V^*,$$

or for compactness we can also write

$$\det A\colon \det V \to \det V^*.$$

We will define

$$\mathrm{Pf}(A) = \frac{1}{n!}\,\omega^{\wedge n} \in \Lambda^{2n} V^* = \det V^*.$$

The reason for the $n!$ will become clear later.

Because $\det V$ and $\det V^*$ are 1-dimensional, we can regard $\det A$ as an element of $(\det V^*)^{\otimes 2}$, so it's reasonable to hope that $\det A = \mathrm{Pf}(A)^2$.

Discuss any questions that you have about this set-up.

2. Let W be another copy of $\mathbb{R}^{2n}$, and let $B\colon W \to V$. Then the 2-form $B^*\omega \in \Lambda^2 W^*$ corresponds to the map

$$B^\top \circ A \circ B\colon W \to V \to V^* \to W^*.$$

As matrices, we know that $\det(B^\top A B) = (\det B)^2 \cdot \det A$. Convince yourselves that this is ok in our invariant set-up, where

$$\begin{aligned} \det B,\, \det B^\top &\in \det W^* \otimes \det V \\ \det A &\in (\det V^*)^{\otimes 2} \\ \det(B^\top A B) &\in (\det W^*)^{\otimes 2}. \end{aligned}$$

Then convince yourselves that

$$\mathrm{Pf}(B^\top \circ A \circ B) = \det B \cdot \mathrm{Pf}(A),$$

because

$$(B^*\omega)^{\wedge n} = B^*(\omega^{\wedge n})$$

and $B^*\colon \Lambda^{2n}V^* \to \Lambda^{2n}W^*$ is $\det B$.

3. In a skew-symmetric analogue of Sylvester's law of inertia, we can choose a basis

$$e_1, e_2, \dots, e_{2n}$$

in which ω becomes

$$e_1^* \wedge e_2^* + \cdots + e_{2m-1}^* \wedge e_{2m}^*$$

for some $m \leq n$, and A becomes

$$\begin{pmatrix} 0 & 1 & & & & & & & \\ -1 & 0 & & & & & & & \\ & & \ddots & & & & & & \\ & & & 0 & 1 & & & & \\ & & & -1 & 0 & & & & \\ & & & & & 0 & & & \\ & & & & & & \ddots & & \\ & & & & & & & & 0 \end{pmatrix}$$

where the number of 2×2 blocks is m and the number of zeros is $2n-2m$. Think about this.

4. View this choice of basis as an isomorphism $B\colon \mathbb{R}^{2n} \to V$.

   If $m < n$ then $(B^*\omega)^{\wedge n} = 0$ and $\det(B^\top AB) = 0$, so $\det A = 0 = (\mathrm{Pf}\, A)^2$.

   If $m = n$ then $(B^*\omega)^{\wedge n}$ is $n!$ times the standard volume form $e_1^* \wedge \cdots \wedge e_{2n}^*$, and $\det(B^\top AB) = 1$, so by #2 we get $\det A = (\det B)^{-2} = (\mathrm{Pf}\, A)^2$.