Worksheet 11

Math 607, Connections and Characteristic Classes

~~Wednesday, February 17, 2021~~
Monday, March 1, 2021

This worksheet builds up to Pontryagin classes.

1. For a real vector bundle E, we can consider the dual bundle E^*.
   (a) Convince yourselves that $E^* \cong E$.
   Hint: Choose a Riemannian metric on E.
   (b) This is also a good time to discuss any lingering questions you have about dual bundles.

2. For a complex vector bundle F, we can consider the dual bundle F^*, the conjugate bundle $\bar{F}$, and the conjugate dual bundle $\bar{F}^*$.
   (a) Convince yourselves that $\bar{F} \cong F^*$, and that $\bar{F}^* \cong F$.
   Hint: Choose a Hermitian metric on F.
   (b) This is also a good time to discuss any lingering questions about "linear in the first argument and conjugate-linear in the second."
   (c) But have seen that if F is the tangent bundle of S^2, regarded as a complex line bundle, then $\int c_1(F) = 2$, so $\int c_1(F^*) = -2$, so $F \ncong F^*$.

   Similarly, if $F = \mathcal{O}(-1)$ is the tautological line bundle on $\mathbb{CP}^1$ then $\int c_1(F) = -1$, so $\mathcal{O}(-1) \ncong \mathcal{O}(1)$. Actually this is a roundabout way to see that $TS^2 \cong \mathcal{O}(2)$.

   Discuss any questions that you have about this.

3. Let E be a real vector bundle, and let $F = E \otimes \mathbb{C}$, which is a complex vector bundle.
   (a) Convince yourselves that $\bar{F} \cong F$ by thinking about transition functions.
   (b) Thus in $H^{2k}(X, \mathbb{C})$ we have
   $$\mathrm{ch}_k(F) = \mathrm{ch}_k(\bar{F}) = \mathrm{ch}_k(F^*) = (-1)^k \,\mathrm{ch}_k(F),$$
   so $\mathrm{ch}_k(F) = 0$ if F is odd, and similarly with c_k. Think about this directly on the level of forms: if we choose a Riemannian metric on E and a compatible connection ∇, then its curvature form Ω takes values in skew-symmetric matrices, so
   $$\mathrm{ch}_k(\nabla) = \tfrac{1}{k!}\left(\tfrac{i}{2\pi}\right)^k \mathrm{tr}(\overbrace{\Omega \wedge \cdots \wedge \Omega}^{k \text{ times}})$$
   vanishes if k is odd.
   (c) We define the Pontryagin classes of a real bundle by $p_k(E) := c_{2k}(F) \in H^{4k}(X, \mathbb{R})$.