Worksheet 12

Math 607, Connections and Characteristic Classes

Wednesday, March 3, 2021

Fix integers $m \geq n > 0$, and identify $\mathbb{R}^{mn}$ with the space of $m \times n$ matrices, or of linear maps $\mathbb{R}^n \to \mathbb{R}^m$. Let $\Phi_k \subset \mathbb{R}^{mn}$ be the space of matrices of rank $\leq k$, and let $\Phi_k^\circ = \Phi_k \setminus \Phi_{k-1}$ be the space of matrices of rank exactly k.

(Everything works exactly the same with $\mathbb{C}$ in place of $\mathbb{R}$.)

1. Observe that $\mathrm{GL}_m \mathbb{R} \times (\mathrm{GL}_n \mathbb{R})^{\mathrm{op}}$ acts on $\mathbb{R}^{mn}$ by letting
$$(A, B) \cdot M = AMB.$$
Or if you don't like the $^{\mathrm{op}}$ you could take AMB^{-1}.

   Convince yourselves that Φ_k° is the orbit of the block matrix
$$\begin{array}{cc} & \begin{array}{cc} k & n-k \end{array} \\ \begin{array}{c} k \\ m-k \end{array} & \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \end{array}$$
Thus in particular Φ_k° is a manifold.

2. Compute the dimension of Φ_k° by finding the stabilizer of the matrix in #1. So you want matrices in $\mathrm{GL}_m \mathbb{R}$ and $\mathrm{GL}_n \mathbb{R}$ such that
$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} E & F \\ G & H \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix},$$
but I leave it to you to work out the size of the blocks, the relations this gives you, and the dimension of the stabilizer.

   The cleanest answer is that Φ_k° has codimension $(m-k)(n-k)$ in $\mathbb{R}^{mn}$.

(Continued on next page.)

3. Let

$$\tilde{\Phi}_k = \Big\{ (M, W) \in \mathbb{R}^{mn} \times \mathrm{Gr}(k, n) \ : \ \mathrm{image}(M) \subset W \Big\}.$$

(a) Observe that $\pi_2 \colon \tilde{\Phi}_k \to \mathrm{Gr}(k, n)$ is a vector bundle of rank nk, so $\tilde{\Phi}_k$ is a smooth manifold.

(b) Observe that the image of $\pi_1 \colon \tilde{\Phi}_k \to \mathbb{R}^{mn}$ is Φ_k, and that it's an isomorphism over Φ_k°, so that gives another way of showing that Φ_k° is a manifold and computing its dimension.

(Recall that the dimension of $\mathrm{Gr}(k, n)$ is $k(n-k)$.)

(The map $\tilde{\Phi}_k \to \Phi_k$ is what's called a resolution of singularities.)

(c) Challenge: Observe that $\mathrm{GL}_m \mathbb{R} \times (\mathrm{GL}_n \mathbb{R})^{\mathrm{op}}$ acts on $\tilde{\Phi}_k$, and that the map $\pi_1 \colon \tilde{\Phi}_k \to \mathbb{R}^{mn}$ is equivariant.

Thus at any point $(M, W) \in \tilde{\Phi}_k$, the derivative of π_1 takes the tangent space of the orbit of (M, W) in $\tilde{\Phi}_k$ surjectively onto the tangent space of the orbit of M in $\mathbb{R}^{mn}$.

Conclude that if a map $f \colon X \to \mathbb{R}^{mn}$ is transverse to all the orbits, then it is transverse to the map π_2, so the fiber product $X \times_{\mathbb{R}^{mn}} \tilde{\Phi}_k$ is a smooth manifold.