Worksheet 12
Math 607, Homological Algebra
Wednesday, April 29, 2020

1. Consider the ring $R = \mathbb{Z}[\sqrt{-3}]$ and the maximal ideal $m = (2, 1 + \sqrt{-3})$. Adapt your Macaulay2 code from last time to study $\text{Tor}_i^R(R/m, R/m)$.

   $R = \mathbb{Z}[y]/(y^2+3)$
   $m = \text{ideal}(2, 1+y)$
   $C = \text{resolution}(R^1/m, \text{LengthLimit} => 6)$

   Have a look at the differentials $C.dd_1$, $C.dd_2$, etc. They should remind you of the 2-periodic resolution from worksheet 7, but now instead of $xz + yw = 0$ you have

   $$2 \cdot 2 + (1 + \sqrt{-3})(-1 + \sqrt{-3}) = 0.$$  

   Continue as you did last time:

   $k = R/m$
   $C' = C * k$

   Have a look at the differentials of $C'$, and conclude that

   $$\text{Tor}_i^R(R/m, R/m) = R/m$$  

   for all $i$. So $R$ has infinite global dimension, unlike $\mathbb{Z}[(\sqrt{-5})$ or $\mathbb{Z}[1+\frac{1}{2}\sqrt{-3}]$.

2. If you have extra time: We studied $\mathbb{Z}[(\sqrt{-5})$ and $\mathbb{Z}[(\sqrt{-3})$, which are flat extensions of $\mathbb{Z}$, and the self-Tors of their quotients by maximal ideals, finding that the first has global dimension 1, but the second has global dimension $\infty$ because of the maximal ideal $(2, 1 + \sqrt{-3})$. Do a similar analysis of $\mathbb{R}[x, y]/(y^2 - x^3 + x)$ and $\mathbb{R}[x, y]/(y^2 - x^3 - x^2)$, which are flat extensions of $\mathbb{R}[x]$. Or use $\mathbb{C}$ coefficients if you prefer.