Worksheet 2

Math 607, Connections and Characteristic Classes

Wednesday, January 7, 2021

Let X be a smooth manifold and $p\colon E \to X$ a smooth vector bundle. We just defined a *connection* on E as a map

$$\nabla\colon \Gamma(TX) \otimes_{\mathbb{R}} \Gamma(E) \to \Gamma(E)$$

which is tensorial in the first argument

$$\nabla_{fV}(\sigma) = f\nabla_V(\sigma)$$

and satisfies a Leibniz rule in the second

$$\nabla_V(f\sigma) = V(f)\sigma + f\nabla_V(\sigma),$$

or equivalently as a map

$$\nabla\colon \Gamma(E) \to \Gamma(\Omega^1_X \otimes E)$$

satisfying

$$\nabla(f\sigma) = df \otimes \sigma + f\nabla\sigma.$$

0. Introduce yourself to your colleague(s). Are they teaching this term?

1. Convince yourselves that the tangent bundle of E sits in an exact sequence

$$0 \to p^*E \to TE \xrightarrow{Dp} p^*TX \to 0,$$

   where Dp is the derivative of p.

   (One of you should draw a picture for everyone to look at. . .)

2. Show that a connection on E determines a splitting of that exact sequence.

3. Challenge: Conversely, given a splitting of that exact sequence, we can try to get a connection, but we need some additional conditions on the splitting. What conditions do we need?