Worksheet 21

Math 607, Homological Algebra

Wednesday, May 27, 2020

1. Sketch the line $x = y = 0$ and the point $x = y = z = 0$ in 3-space.

2. Let $R = k[x, y, z]$. Using the Koszul resolution
\[
0 \rightarrow R \rightarrow R^3 \rightarrow R^3 \rightarrow R \rightarrow R/(x, y, z) \rightarrow 0,
\]
verify that
\[
\text{Ext}^i_R(R/(x, y, z), R) = \begin{cases} 
0 & i = 0 \\
0 & i = 1 \\
0 & i = 2 \\
R/(x, y, z) & i = 3.
\end{cases}
\]

3. For a finite resolution
\[
0 \rightarrow A^{-n} \rightarrow \cdots \rightarrow A^{-1} \rightarrow A^0 \rightarrow N \rightarrow 0,
\]
we have a second-quadrant spectral sequence
\[
E_1^{p,q} = \text{Ext}^q_R(M, A^p) \Rightarrow \text{Ext}^{p+q}(M, N).
\]
Using the Koszul resolution
\[
0 \rightarrow R \xrightarrow{(y, -x)} R^2 \xrightarrow{(x, y)} R \rightarrow R/(x, y) \rightarrow 0,
\]
use this Koszul resolution to show that
\[
\text{Ext}^i_R(R/(x, y, z), R/(x, y)) = \begin{cases} 
0 & i = 0 \\
k & i = 1 \\
k^2 & i = 2 \\
k & i = 3,
\end{cases}
\]
where $k$ is short for $R/(x, y, z)$.

Hint: The differentials on the $E_1$-page are what you think they are.

4. To compute $\text{Ext}^i_R(R/(x, y, z), R/(x, y))$ directly, you could have applied $\text{Hom}(\cdot, R/(x, y))$ to the Koszul resolution in problem 2. Observe that this would have been harder.