Let $V = \mathbb{R}^n$ or $\mathbb{C}^n$. The point of this worksheet is to understand that the tangent bundle of $\text{Gr}(k, n) = \text{Gr}(k, V)$ is naturally identified with

$$\mathcal{H}om(S, Q) = S^* \otimes Q,$$

where $S$ and $Q$ are the tautological bundles.

In lecture we described $\text{Gr}(k, n)$ as $G/H$, where $G = \text{GL}_n$ and $H$ is the subgroup of block upper-triangular matrices

$$\begin{pmatrix}
* & \cdots & * & * & \cdots & * \\
\vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
* & \cdots & * & * & \cdots & * \\
0 & \cdots & 0 & * & \cdots & * \\
\vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & * & \cdots & * 
\end{pmatrix}$$

where the first blocks is $k \times k$ and the second is $(n-k) \times (n-k)$. Alternatively, $H$ is the stabilizer of the subspace $W_0 \subset V$ spanned by $e_1, \ldots, e_k$. So we have the quotient map $q: G \to \text{Gr}(k, n)$ defined by $q(A) = A \cdot W_0$, and the fiber $q^{-1}(W_0)$ is $H$. The group $G$ is an open subset of $\text{Hom}(V, V)$, so its tangent space at every point is $\text{Hom}(V, V)$. The natural map

$$\text{Hom}(V, V) \to \text{Hom}(W_0, V/W_0)$$

reads off the bottom left $(n-k) \times k$ block, so if $K \subset \text{Hom}(V, V)$ is the kernel of this map then $H = G \cap K$, so the tangent space of $H$ at every point is $K$.

Considering the exact sequence

$$0 \to TH \to TG \to q^*T\text{Gr} \to 0$$

of vector bundles on $G$, it sure looks like $T\text{Gr} = \mathcal{H}om(S, Q)$, although we have to think a bit about what $TH$ means. Can you turn these ideas into a complete proof?