Worksheet 3

Math 607, Connections and Characteristic Classes

Wednesday, January 13, 2021

Let $V = \mathbb{R}^n$ or $\mathbb{C}^n$. The point of this worksheet is to understand that the tangent bundle of $\mathrm{Gr}(k,n) = \mathrm{Gr}(k,V)$ is naturally identified with

$$\mathcal{H}om(S,Q) = S^* \otimes Q,$$

where S and Q are the tautological bundles.

In lecture we described $\mathrm{Gr}(k,n)$ as G/H, where $G = \mathrm{GL}_n$ and H is the subgroup of block upper-triangular matrices

$$\begin{pmatrix} * & \cdots & * & * & \cdots & * \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ * & \cdots & * & * & \cdots & * \\ 0 & \cdots & 0 & * & \cdots & * \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & * & \cdots & * \end{pmatrix}$$

where the first blocks is $k \times k$ and the second is $(n-k)\times(n-k)$. Alternatively, H is the stabilizer of the subspace $W_0 \subset V$ spanned by $e_1, \dots, e_k$. So we have the quotient map $q\colon G \to \mathrm{Gr}(k,n)$ defined by $q(A) = A \cdot W_0$, and the fiber $q^{-1}(W_0)$ is H. The group G is an open subset of $\mathrm{Hom}(V,V)$, so its tangent space at every point is $\mathrm{Hom}(V,V)$. The natural map

$$\mathrm{Hom}(V,V) \twoheadrightarrow \mathrm{Hom}(W_0,\, V/W_0)$$

reads off the bottom left $(n-k)\times k$ block, so if $K \subset \mathrm{Hom}(V,V)$ is the kernel of this map then $H = G \cap K$, so the tangent space of H at every point is K.

Considering the exact sequence

$$0 \to TH \to TG \to q^* T\,\mathrm{Gr} \to 0$$

of vector bundles on G, it sure looks like $T\,\mathrm{Gr} = \mathcal{H}om(S,Q)$, although we have to think a bit about what TH means. Can you turn these ideas into a complete proof?