Worksheet 5

Math 607, Connections and Characteristic Classes

Monday, January 25, 2021

Let \( \phi : \mathbb{R}^2 \to S^2 \subset \mathbb{R}^3 \) be the parametrization of \( S^2 \) minus the north pole given by stereographic projection:

\[
\phi(x, y) = \left( \frac{2x}{1 + x^2 + y^2}, \frac{2y}{1 + x^2 + y^2}, \frac{x^2 + y^2 - 1}{1 + x^2 + y^2} \right).
\]

I don’t think you’ll need the inverse map, but just in case, it’s given by

\[
\psi(u, v, w) = \left( \frac{u}{1 - w}, \frac{v}{1 - w} \right).
\]

1. Convince yourselves that these formulas are correct.

2. Compute \( \phi^* \left( \frac{\partial}{\partial x} \right) \) and \( \phi^* \left( \frac{\partial}{\partial y} \right) \). Concretely, these should be triples of functions on \( \mathbb{R}^2 \).

3. Let \( \nabla \) be the trivial connection on the trivial bundle \( \phi^* \mathbb{R}^3 \). Compute \( \nabla \left( \frac{\partial}{\partial x} \right) \) and \( \nabla \left( \frac{\partial}{\partial y} \right) \).

Concretely, if \( \phi^* \left( \frac{\partial}{\partial x} \right) \) is a triple of functions on \( \mathbb{R}^2 \), just take the exterior derivative of each component to get a triple of 1-forms on \( \mathbb{R}^2 \). And similarly with \( \phi^* \left( \frac{\partial}{\partial y} \right) \).

4. Let \( \nabla' \) be the induced connection on \( TS^2 \), obtained from \( \nabla \) by orthogonally projecting. Compute \( \nabla' \left( \frac{\partial}{\partial x} \right) \) and \( \nabla' \left( \frac{\partial}{\partial y} \right) \).

5. As a sanity check, make sure that your connection is symmetric:

\[
\nabla' \left( \frac{\partial}{\partial x} \right) - \nabla' \left( \frac{\partial}{\partial y} \right) = \left[ \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right] = 0,
\]

that is,

\[
\nabla' \left( \frac{\partial}{\partial y} \right) = \nabla' \left( \frac{\partial}{\partial x} \right).
\]