Worksheet 5

Math 607, Connections and Characteristic Classes

Monday, January 25, 2021

Let $\phi\colon \mathbb{R}^2 \to S^2 \subset \mathbb{R}^3$ be the parametrization of S^2 minus the north pole given by stereographic projection:

$$\phi(x,y) = \left(\frac{2x}{1+x^2+y^2}, \frac{2y}{1+x^2+y^2}, \frac{x^2+y^2-1}{1+x^2+y^2}\right).$$

I don't think you'll need the inverse map, but just in case, it's given by

$$\psi(u,v,w) = \left(\frac{u}{1-w}, \frac{v}{1-w}\right).$$

1. Convince yourselves that these formulas are correct.

2. Compute $\phi_*(\frac{\partial}{\partial x})$ and $\phi_*(\frac{\partial}{\partial y})$. Conceretely, these should be triples of functions on $\mathbb{R}^2$.

3. Let ∇ be the trivial connection on the trivial bundle $\phi^*T\mathbb{R}^3$. Compute

$$\nabla(\tfrac{\partial}{\partial x}) \qquad \text{and} \qquad \nabla(\tfrac{\partial}{\partial y}).$$

   Conceretely, if $\phi_*(\frac{\partial}{\partial x})$ is a triple of functions on $\mathbb{R}^2$, just take the exterior derivative of each component to get a triple of 1-forms on $\mathbb{R}^2$. And similarly with $\phi_*(\frac{\partial}{\partial y})$.

4. Let ∇' be the induced connection on TS^2, obtained from ∇ by orthogonally projecting. Compute

$$\nabla'(\tfrac{\partial}{\partial x}) \qquad \text{and} \qquad \nabla'(\tfrac{\partial}{\partial y}).$$

5. As a sanity check, make sure that your connection is symmetric:

$$\nabla'_{\frac{\partial}{\partial x}}(\tfrac{\partial}{\partial y}) - \nabla'_{\frac{\partial}{\partial y}}(\tfrac{\partial}{\partial x}) = [\tfrac{\partial}{\partial x}, \tfrac{\partial}{\partial y}] = 0,$$

   that is,

$$\nabla'_{\frac{\partial}{\partial x}}(\tfrac{\partial}{\partial y}) = \nabla'_{\frac{\partial}{\partial y}}(\tfrac{\partial}{\partial x}).$$