Worksheet 7

Math 607, Connections and Characteristic Classes

Friday, February 5, 2021

This will take more than one day, so keep your answers for next time.

Let $X \subset \mathbb{R}^3$ be the graph of a function: $z = f(x, y)$. Thus X is a surface, parametrized by

$$\phi\colon \mathbb{R}^2 \to \mathbb{R}^3 \qquad\qquad \phi(x, y) = (x, y, f(x, y))$$

Eventually we're going to assume that $f(0, 0) = 0$ and $f_x(0, 0) = f_y(0, 0) = 0$, so the surface passes through the origin and the tangent plane there is the xy-plane. For any surface in $\mathbb{R}^3$ and any point on that surface, we can change coordinates on $\mathbb{R}^3$ to put the point at the origin and make the tangent plane the xy-plane, so this is no loss of generality.

Last time we used the notation $\frac{\partial}{\partial x}$ all over the place, but for compactness let's use ∂_x this time, and instead of $\frac{\partial f}{\partial x}$ let's use f_x.

0. Draw a little picture to put yourselves in the right frame of mind.

1. Compute $\phi_*(\partial_x)$ and $\phi_*(\partial_y)$.

2. Let g be the standard Riemannian metric in $\mathbb{R}^3$ – the good old dot product – and let $g' = \phi^* g$. Compute the matrix of g' in the basis ∂_x, ∂_y. Compute the volume form, or since we're on a surface you could just call it the area form:

$$dA = \sqrt{\det g'}\, dx \wedge dy.$$

3. View $\left(\begin{smallmatrix} 0 \\ 0 \\ 1 \end{smallmatrix}\right)$ as a section of $T\mathbb{R}^3|_X$. Find its orthogonal projection onto TX, expressed in terms of ∂_x and ∂_y.

4. As before, we'll take the standard connection ∇ on $T\mathbb{R}^3|_X$, and hit it with the orthogonal projection $T\mathbb{R}^3|_X \to TX$ to get the Levi-Civita connection ∇' on X.

   Compute

$$\nabla'_x \partial_x \qquad \nabla'_x \partial_y \qquad \nabla'_y \partial_x \qquad \nabla'_y \partial_y,$$

   where ∇'_x is short for ∇'_{∂_x}.

5. Compute

$$\nabla'_x \nabla'_y \partial_x - \nabla'_y \nabla'_x \partial_x \qquad \nabla'_x \nabla'_y \partial_y - \nabla'_y \nabla'_x \partial_y$$

   at the point $(0,0)$, assuming that $f_x(0,0) = f_y(0,0) = 0$. This an exercise plugging in 0 as early as possible, but not earlier.

6. Suppose we have the graph of a function of one variable, $y = h(x)$, and that $h(0) = h'(0) = 0$. Convince yourselves that the radius of the osculating circle at the origin is $|1/h''(0)|$. Draw a little picture.

7. If $(a, b, 0) \in \mathbb{R}^3$ is a unit tangent vector to our surface X at the origin, we define the (signed) curvature in that direction to be 1 over the (signed) radius of the osculating circle of the curve obtained by slicing X with the plane spanned by $(a, b, 0)$ and the normal vector $(0, 0, 1)$. Convince yourselves that it equals

$$a^2 f_{xx}(0,0) + 2ab f_{xy}(0,0) + b^2 f_{yy}(0,0).$$

8. That was the quadratic form associated to the symmetric matrix

$$\begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix}.$$

   Convince yourselves that if the signed curvature is maximized in one direction and minimized in another, then the product of those two curvatures is the determinant of the matrix:

$$f_{xx} f_{yy} - f_{xy}^2.$$

   This should be the the Pfaffian of the curvature matrix that you found earlier.